UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 7377	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center 6b. OFFICE SYMBOL *(If applicable)*	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS *(City, State, and ZIP Code)* New London Laboratory New London, Connecticut 06320	7b. ADDRESS *(City, State, and ZIP Code)*
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Naval Material Command 8b. OFFICE SYMBOL *(If applicable)*	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS *(City, State, and ZIP Code)* Washington, DC 20360	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO.: PROJECT NO.: ZR0000101 TASK NO.: WORK UNIT ACCESSION NO.:

11. TITLE *(Include Security Classification)* EVALUATION OF DENSITIES AND DISTRIBUTIONS VIA HERMITE AND GENERALIZED LAGUERRE SERIES EMPLOYING HIGH-ORDER EXPANSION COEFFICIENTS DETERMINED RECURSIVELY VIA MOMENTS OR CUMULANTS

12. PERSONAL AUTHOR(S)
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13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT *(Year, Month, Day)*	15. PAGE COUNT
	FROM ____ TO ____	85 2 25	

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
Characteristic Function Expansion
Cumulative Distribution Function
Exceedance Distribution Function

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

High-order series expansions of probability density functions and cumulative distribution functions, in Hermite as well as generalized Laguerre orthogonal polynomials, have been obtained, where the weighting functions in both cases can have arbitrary (mismatched) parameter values; that is, the two free parameters α and β in the weightings

$$w(u) = \frac{1}{\sqrt{2\pi}\,\beta} \exp\left(-\frac{(u-\alpha)^2}{2\beta^2}\right) \text{ for all } u, \qquad \text{Hermite}$$

$$w(u) = \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \text{ for } u > 0, \qquad \text{generalized Laguerre}$$

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Albert H. Nuttall	22b. TELEPHONE *(Include Area Code)* (203) 440-4618 / 22c. OFFICE SYMBOL

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

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18. (Cont'd)

False Alarm/Detection Probability
Generalized Laguerre Expansions
Hermite Expansions
High-Order Moments/Cumulants
Mismatched Parameter Values
Probability Density Function
Recursive Expansion Coefficients
Recursive Moments/Cumulants
Rician Variate
Shot Noise

19. (Cont'd)

need not be chosen so that the first two expansion coefficients b_1 and b_2 in the ortho-normal series are zero. (The zero-th order expansion coefficient b_0 is never zero.) Nonetheless all the available N lowest-order moments of the approximating probability density function are maintained identical to those of the given probability density function, regardless of the weighting employed and any of its free parameter values.

It has been discovered that deliberate mismatch of α and β results in faster-decaying coefficient sequences $\{b_n\}_0^N$ than when α and β are chosen to make $b_1 = b_2 = 0$, which is a common choice. For example, the central limit theorem is just such a case, where α and β in the Hermite expansion are taken as the mean and standard deviation, respectively, and the number of moments employed is limited to just order $N = 2$.

A fast trial-and-error procedure is used in general to determine good values of weighting parameters α and β. The only statistics needed about the given probability density function or cumulative distribution function are either its moments or cumulants, through order N. Furthermore, all the results presented actually apply to functions which have arbitrary area (not necessarily equal to unity) and to functions which can become negative. In fact, one of the applications considered is to a shot noise process where the continuous part of the probability density function has area less than 1, and which is well approximated by a generalized Laguerre series expansion.

The high-order expansion coefficients for both the Hermite and generalized Laguerre series can each be obtained by any one of three fast recursive procedures (all of which have been programmed, and for which program listings are presented):

(a) recursively via cumulants,

(b) directly via moments,

(c) recursively via moments.

The forms of these three recursive procedures differ in the Hermite vs. Laguerre cases; however, they are basically either convolutions or finite alternating series with binomial coefficients. The occurrence and quantitative value of round-off error for large N is easily discerned in a plot of the expansion coefficient sequence for each choice of α and β, and for each of the three procedures, as well as for both types of series expansions.

Comparisons of the accuracy of the three alternative recursive procedures reveals that expansion coefficients determined recursively via cumulants are generally most accurate and least susceptible to round-off error. Numerous examples of series expansions of probability density functions and cumulative distribution functions are given, including one with $N = 150$ terms, where the last expansion coefficient is of size 1E-10 relative to the leading coefficient b_0. Estimates of the error associated with the approximations obtained by the Hermite and generalized Laguerre series are derived and compared with results of several examples.

NUSC Technical Report 7377
28 February 1985

Evaluation of Densities and Distributions via Hermite and Generalized Laguerre Series Employing High-Order Expansion Coefficients Determined Recursively via Moments or Cumulants

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PREFACE

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 33), and under NUSC Project No. A65090, Subproject No. ZR000-01, "Evaluation of Incoherent Field in the Arctic Environment," Principal Investigator R. L. Deavenport (Code 3332), Program Manager Gary Morton, Naval Material Command (MAT 05).

The Technical Reviewer for this report was Dr. Roy L. Streit (Code 3232).

Reviewed and Approved: 28 February 1985

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LIST OF SYMBOLS

$w(u)$	Weighting function at argument u
α,β	Parameters of weighting w; (1),(2)
b_n	n-th expansion coefficient in orthonormal polynomial series
N	Number of terms employed in series expansion
p	Probability density function
P	Cumulative distribution function; integral of p
$1-P$	Exceedance distribution function
f	Characteristic function corresponding to p
μ_n	n-th moment of p or f; (3)
χ_n	n-th cumulant of p or f; (7)
ν_n	n-th moment of weighting w; (11)
Q_n	n-th order orthonormal polynomial
p_N, P_N	N-th order approximations to p,P
E_N	Weighted squared error
A	Area of p; (24)
M	Mean location of p; (24)
R	Rms width of p; (24)
γ,ω	Parameters of probability density function p
ϕ	Normalized Gaussian function; (41)
Φ	Integral of ϕ; (41)
He_n	n-th Hermite polynomial; (50)
a_n, c_n	Auxiliary expansion coefficients; (44)-(47) or (90)-(92)
w,z	Auxiliary variables

LIST OF SYMBOLS (Cont'd)

$\hat{\chi}_n$	Normalized cumulants; (62)
$\hat{\mu}_n$	Normalized moments; (69)
$\tilde{\mu}_n$	Alternative normalized moments; (118)
$\hat{He}_n$	n-th normalized Hermite polynomial; (68)
Hi_n	n-th modified Hermite polynomial; (74)
$\hat{Hi}_n$	n-th normalized modified Hermite polynomial; (80)
$L_n^{(\alpha)}$	n-th generalized Laguerre polynomial; (96)
${}_1F_1$	Confluent hypergeometric function
d_m	Auxiliary variables in (112)
RC	Recursively via Cumulants
DM	Directly via Moments
RM	Recursively via Moments
$\mathcal{J}, \theta$	Parameters of p in (141)
Q_ν	Generalized Q-function; (165)
P_0	Area of impulse at origin of p, for shot noise
F	Hypergeometric function ${}_2F_1$
Env	Envelope function
$E_n(u;p)$	Estimated error of p_n; (178) or (183)
$E_n(u;P)$	Estimated error of P_n; (180) or (185)

EVALUATION OF DENSITIES AND DISTRIBUTIONS VIA HERMITE AND GENERALIZED LAGUERRE SERIES EMPLOYING HIGH-ORDER EXPANSION COEFFICIENTS DETERMINED RECURSIVELY VIA MOMENTS OR CUMULANTS

INTRODUCTION

In the theoretical analysis of performance of some systems with nonlinearities and/or memory, it often happens that the only statistics about the decision (or output) random variable of interest that can be easily found are the moments, or in other cases, the cumulants. Explicit relations for the low-order expansion coefficients in Edgeworth or Gram-Charlier series are available in terms of the available moments or cumulants [1, pp. 172 and 191], [2, pp. 223 and 226], [3, pp. 157 and 159]. However, for higher-order moments and cumulants, these explicit nonrecursive relations are very tedious to derive, become extremely lengthy, and are not practical to use.

We will address the problem of obtaining accurate high-order series expansion approximations of the probability density function and cumulative distribution function of a random variable of interest, in terms of the available moments or cumulants of that random variable. The necessity of being able to approximate probability density functions and cumulative distribution functions from knowledge of either the moments or the cumulants, is that some physical problems have these particular statistics as natural and convenient starting points. For example, if a physical processor sums together a number of independent Rician random variates, the characteristic function and cumulants of the individual random variables or their sum are not available in any useful analytic form; however, the high-order moments of an individual Rician variate can be easily and accurately evaluated by recurrence, and thereby the moments of the sum can be obtained. Conversely, for shot noise with random amplitude and duration modulation, the probability density function is not readily available, whereas the characteristic function is, and the cumulants are simple to evaluate [4, appendix C].

The particular series expansions we employ are based on the two special classes of weighting functions

$$w(u) = \frac{1}{(2\pi)^{1/2}\beta}\exp\left(-\frac{(u-\alpha)^2}{2\beta^2}\right) \quad \text{for all } u \qquad \text{Hermite} \tag{1}$$

and

$$w(u) = \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \quad \text{for } u > 0 \qquad \text{generalized Laguerre.} \tag{2}$$

The orthonormal polynomials associated with these weightings are directly related to the Hermite and generalized Laguerre polynomials, respectively [5, 22.2.15 and 22.2.13]. The weightings each have two free parameters, α and β, which can be manipulated to advantage in obtaining finite (high-order) series expansions which well approximate a given (unknown) probability density function and cumulative distribution function.

The question of when a set of moments uniquely determines the probability density function is a difficult one; see, for example, [3, pp. 109-112 and 179]. Also, the convergence of the series is very involved [2, pp. 223 and 258], [3, pp. 161-163]. But, even if the series is divergent, use of a limited number of expansion coefficients often gives a satisfactory approximation to the desired probability density function [3, p. 167]. We presume here that the moments do uniquely determine the probability density function and are buoyed in that respect by the comment [3, p. 87] that most distributions in statistical practice do possess this property.

The main idea in the series expansion approach here is not necessarily to get as many terms as possible, but rather to get as rapid convergence as possible of the series. If a particular choice of weighting parameters α and β results in sufficiently small expansion coefficients, say, at order 10, this is better than another choice of α and β where 20 or 30 terms are required for the same size coefficients. In fact, if α and β could be chosen such that the series terminated (zero coefficients) after a few terms, that would be ideal; however, this is not the case, and in fact, the choice of α and β requires some trial-and-error to achieve rapidly decreasing coefficients.

The expansion coefficients of a given probability density function, in an orthonormal set of Hermite or generalized Laguerre polynomials, are denoted by $\{b_n\}_0^N$, where N is the number of available or known moments or cumulants. Very often, the choice of α and β in (1) or (2) has been made such that $b_1 = 0$ and $b_2 = 0$, for purposes of analytic simplicity and for hopeful early termination of the series; see for example [1, pp. 171 and 191], [2, p. 223], [3, p. 159]. However, it will be demonstrated that this is generally not the best choice, and that more rapidly decaying coefficients can be achieved by other (mismatched) values of α and β, which must be searched for numerically; this possibility is also mentioned in [3, p. 164]. In fact, an example will be given which illustrates that the choice of parameters α and β to make expansion coefficients b_1 and b_2 zero, can in fact, lead to a divergent Hermite series.

Depending on the available information about the probability density function, i.e., moments or cumulants, a variety of methods will be given for determining the expansion coefficients $\{b_n\}$. In particular, for both the Hermite and generalized Laguerre series, we can get the coefficients by three different procedures:

(a) recursively via cumulants,

(b) directly via moments,

(c) recursively via moments.

The reason for having these alternatives is that the calculation of expansion coefficients $\{b_n\}$ for high-order n invariably runs into large round-off error. In order to reduce this round-off error, the amount of number-crunching on the computer should be minimized, and any spurious transformations between moments and cumulants should be avoided if possible. Thus it is desireable to have techniques which can accomplish the desired goal of evaluating expansion coefficients $\{b_n\}$ as directly as possible from the available information. The use of different alternatives also enables comparisons of the computed expansion coefficients and thereby furnishes quantitative assessment of the amount of round-off error. Recursive inter-relationships between moments, central moments, and cumulants are given in [6], including cases of two dependent random variables.

FUNDAMENTAL EQUATIONS

DEFINITION OF STATISTICS

Suppose a function p has known moments*

$$\mu_n = \int du\ u^n\ p(u) \quad \text{for } 0 \le n \le N\ . \tag{3}$$

The function p need not have unit area, i.e., $\mu_0 \neq 1$ is allowed, and p can become negative at some arguments u. Nevertheless, for convenience, and since most of our applications are to random variables, we shall refer to p as a probability density function, and to its running integral

$$P(u) = \int_{-\infty}^{u} dt\ p(t) \tag{4}$$

as a cumulative distribution function. We shall presume that $\mu_0 > 0$ in all cases.

The characteristic function corresponding to probability density function p is the Fourier transform

$$f(i\xi) = \int du\ \exp(i\xi u)\ p(u)\ . \tag{5}$$

When f is expanded in a power series, the result is

$$f(i\xi) = \sum_{n=0}^{\infty} \mu_n (i\xi)^n/n! \tag{6}$$

* Integrals without limits are over the range of nonzero integrand.

in terms of the moments in (3). Alternatively, if $\ln f$ is expanded in a power series,

$$\ln f(i\xi) = \sum_{n=0}^{\infty} \chi_n \, (i\xi)^n/n! \; , \tag{7}$$

where the quantities $\{\chi_n\}$ are the cumulants of p or f. Observe that generally, to the lowest three orders,

$$\chi_0 = \ln f(0) = \ln \mu_0 \neq 0 \; ,$$

$$\chi_1 = \frac{\mu_1}{\mu_0} \; ,$$

$$\chi_2 = \frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2 \; . \tag{8}$$

The available information on probability density function p will be either

$$\text{moments } \{\mu_n\}_0^N \quad \text{or cumulants } \{\chi_n\}_0^N \; . \tag{9}$$

Whichever is available, we wish to get high-order accurate approximations to p and cumulative distribution function P in (4); that is, values of N in the order of 10 to 100 are of interest.

WEIGHTING FUNCTION PROPERTIES

We select a nonnegative weighting function w such that

$$w(u) > 0 \text{ at least where } p(u) \neq 0 \; . \tag{10}$$

We also disallow any impulses in w. The moments of weighting w are defined analogously to (3) as

$$\nu_n = \int du \; u^n \; w(u) \quad \text{for } n \geq 0 \; ; \tag{11}$$

it is presumed that these quantities can be evaluated for as large n as required.

Suppose weighting w has r free parameters (plus a scaling parameter). It might then seem beneficial to choose them such that the moments of w and p are approximately equal,

$$\nu_n \cong \mu_n \quad \text{for } 1 \leq n \leq r \quad (\text{plus } \nu_0 \cong \mu_0) \; , \tag{12}$$

for then the abscissa scales of w and p would tend to match. However, (12) will turn out to be not so desireable, and the choice of the r weighting parameter values should be based on another criterion. The ordinate scale of w is actually immaterial, since the expansion coefficients $\{b_n\}$ will absorb this scaling; so henceforth we presume that $\nu_0 = 1$ with no loss of generality.

APPROXIMATION PROCEDURE

Let Q_n be any n-th order polynomial, and approximate probability density function p by function

$$p_N(u) \equiv w(u) \sum_{n=0}^{N} b_n Q_n(u) \quad \text{where } w(u) > 0 \;, \tag{13}$$

where $\{b_n\}_0^N$ are the expansion coefficients. Define weighted squared error

$$E_N = \int du\ \gamma(u)\ [p(u)-p_N(u)]^2 =$$

$$= \int du\ \gamma(u)\ [p(u)-w(u) \sum_{n=0}^{N} b_n Q_n(u)]^2 \;, \tag{14}$$

where error-weighting γ is nonnegative. If we minimize E_N by choice of expansion coefficients $\{b_n\}_0^N$, there follows the set of linear equations

$$\sum_{n=0}^{N} b_n \int du\ \gamma(u)\ w^2(u)\ Q_k(u)\ Q_n(u) = \int du\ \gamma(u)\ w(u)\ p(u)\ Q_k(u) \quad \text{for } 0 \le k \le N. \tag{15}$$

In order to use only the available information in (9) about p, the right-hand side of (15) must simplify according to the selection

$$\gamma(u) = \frac{K}{w(u)} \quad \text{where } w(u) > 0 \text{ (and arbitrary elsewhere).} \tag{16}$$

Furthermore, since constant K merely scales error E_N, and appears on both sides of (15), we can set K = 1 without loss of generality. Then (14) becomes

$$E_N = \int du\ w(u)\left[\frac{p(u)}{w(u)} - \sum_{n=0}^{N} b_n\ Q_n(u)\right]^2 \quad \text{where } w(u) > 0\ , \tag{17}$$

and (15) reduces to

$$\sum_{n=0}^{N} b_n \int du\ w(u)\ Q_k(u)\ Q_n(u) = \int du\ p(u)\ Q_k(u) \quad \text{for } 0 \le k \le N. \tag{18}$$

In general, this is N+1 simultaneous linear equations in the N+1 unknowns $\{b_n\}_0^N$. The choice $Q_k(u) = u^k$ would lead to an apparently simple set of equations, when (11) and (3) are used. However, a few numerical examples quickly reveals that they are very ill-conditioned, due to the character of the nondiagonal matrix with elements

$$\int du\ w(u)\ Q_k(u)\ Q_n(u) \quad \text{for } 0 \le k,n \le N \tag{19}$$

that appears on the left-hand side of (18). In order to avoid the significant round-off error associated with solving such a system for large N, we choose $\{Q_n\}_0^N$ to be a set of <u>orthonormal</u> polynomials with respect to weighting w; i.e., (19) is 1 for k = n, and 0 otherwise. Also recall that $\nu_0 = \int du\ w(u) = 1$ without loss of generality.

Equation (18) then reduces to an explicit relation for the expansion coefficients:

$$b_k = \int du\ p(u)\ Q_k(u) \quad \text{for } 0 \le k \le N\ , \tag{20}$$

and (17) for the error becomes merely

$$E_N = \int du\ \frac{p^2(u)}{w(u)} - \sum_{n=0}^{N} b_n^2\ . \tag{21}$$

It will be presumed that the integral in (21) is finite; otherwise, the error would be infinite, which is a meaningless problem. This will put some restrictions on the parameter choices of weighting w, since this error integral depends on these parameters as well as on the given probability density function p. The sum of squares in (21) must then be bounded, and in fact affords a measure of the adequacy of approximation (13), by saturating (at an apriori unknown value) for large N.

As N increases, the values of the lower-order expansion coefficients $\{b_k\}$ in (20) do not change. Therefore they only have to be computed once and do not have to be revised as more terms are added in series approximation (13), i.e., larger N.

EQUALITY OF PROBABILITY DENSITY FUNCTION MOMENTS

A very important property of expansion (13) is obtained as follows:

$$\int du\ Q_k(u)\ p_N(u) = \int du\ Q_k(u)\ w(u) \sum_{n=0}^{N} b_n\ Q_n(u) =$$

$$= b_k = \int du\ Q_k(u)\ p(u) \quad \text{for } 0 \le k \le N\ , \tag{22}$$

where we used, in order, (13), the orthonormality of (19), and (20). But since Q_k is a k-th order polynomial, relation (22) states that approximation p_N has exactly the same moments as given probability density function p, from order 0 through order N. This matching of moments between probability density functions p_N and p has been achieved regardless of the weighting w and its particular parameter values. Furthermore, (22) holds independently of whether the weighting-moment equalities in (12) are satisfied or not.

The cumulative distribution function corresponding to approximation p_N is defined as

$$P_N(u) \equiv \int_{-\infty}^{u} dt\ p_N(t) = \sum_{n=0}^{N} b_n \int_{-\infty}^{u} dt\ w(t)\ Q_n(t)\ . \tag{23}$$

Its utility depends on getting closed forms and simple recursions for the general integral on the right-hand side.

PARAMETERS OF GIVEN PROBABILITY DENSITY FUNCTION p

The moments of p were defined in (3). It is useful to define three important parameters of p:

$$\text{Area } A = \int du\ p(u) = \mu_0 \quad (\mu_0 > 0, \text{ but need not be } 1);$$

$$\text{Mean Location } M = \frac{\int du\ u\ p(u)}{\int du\ p(u)} = \frac{\mu_1}{\mu_0}\ ;$$

$$\text{RMS Width } R = \left[\frac{\int du(u-M)^2 p(u)}{\int du\ p(u)}\right]^{1/2} = \left[\frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2\right]^{1/2} . \tag{24}$$

(Conversely, $\mu_0 = A$, $\mu_1 = AM$, $\mu_2 = A(M^2+R^2)$.) These parameters depend on the probability density function p that we are trying to approximate and can be computed from the available information (9). They are useful for determining where the major concentration of p(u) lies on the u-scale, and have obvious physical interpretations.

In terms of the cumulants of p defined in (5)-(8), we have the alternative expressions

$$A = \exp(\chi_0), \qquad M = \chi_1, \qquad R = \chi_2^{1/2} , \tag{25}$$

or conversely

$$\chi_0 = \ln \mu_0 = \ln A, \quad \chi_1 = \frac{\mu_1}{\mu_0} = M, \quad \chi_2 = \frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2 = R^2 . \tag{26}$$

GENERAL RESULTS FOR THREE LOWEST-ORDER POLYNOMIALS Q_n

The weighting function w and associated orthonormal polynomials satisfy the following equation:

$$\int du\; w(u)\; Q_k(u)\; Q_n(u) = \delta_{kn} . \tag{27}$$

Also we have weighting moments

$$\nu_n = \int du\; u^n\; w(u), \quad \text{with } \nu_0 = 1 . \tag{28}$$

It is then a straightforward matter to evaluate the three lowest-order orthonormal polynomials:

$$Q_0(u) = 1 ,$$

$$Q_1(u) = \frac{1}{D_1}(u-\nu_1) ,$$

$$Q_2(u) = \frac{1}{D_2}\left[u^2(\nu_2-\nu_1^2) - u(\nu_3-\nu_2\nu_1) + (\nu_3\nu_1-\nu_2^2)\right] , \tag{29}$$

where

$$D_1 = (\nu_2 - \nu_1^2)^{1/2} ,$$

$$D_2 = (\nu_2 - \nu_1^2)^{1/2} \left[(\nu_4 - \nu_2^2)(\nu_2 - \nu_1^2) - (\nu_3 - \nu_2\nu_1)^2\right]^{1/2} . \tag{30}$$

The general expansion coefficients in (20) then become

$$b_0 = \mu_0 ,$$

$$b_1 = \frac{1}{D_1}(\mu_1 - \nu_1\mu_0) ,$$

$$b_2 = \frac{1}{D_2}\left[\mu_2(\nu_2 - \nu_1^2) - \mu_1(\nu_3 - \nu_2\nu_1) + \mu_0(\nu_3\nu_1 - \nu_2^2)\right] . \tag{31}$$

All these results above are general and make no presumption about weighting moment equalities such as (12).

SPECIAL CHOICES OF WEIGHTING PARAMETERS

Suppose that weighting w has free parameters that can be varied so as to make the mean locations of w and p coincide (see (24)); that is,

$$\text{let} \quad \nu_1 = \frac{\mu_1}{\mu_0} . \tag{32A}$$

(The reason for the discrepancy with (12) is that we have set $\nu_0 = 1$ but have allowed $\mu_0 \neq 1$.) Inspection of (31) gives the following:

$$\text{then} \quad b_1 = 0 \quad \text{and} \quad b_2 = -\frac{\mu_0}{D_2}\left(\nu_2 - \frac{\mu_2}{\mu_0}\right)\left(\nu_2 - \frac{\mu_1^2}{\mu_0^2}\right). \tag{32B}$$

Conversely, (31) shows that requiring $b_1 = 0$ forces the choice in (32A) for ν_1. Thus equality of the first weighting moment ν_1 of w with the first (normalized) moment of probability density function p implies (and is implied by) the vanishing of the first expansion coefficient b_1. This may or may not be a useful choice, but, whether adopted or not, has no bearing on the equality of probability density function moments already demonstrated in (22).

As a second special choice, suppose that weighting w has enough free parameters that we can vary, so as to make the mean locations and rms widths of w and p coincide (see (24)); that is

$$\text{let} \quad \nu_1 = \frac{\mu_1}{\mu_0} \quad \text{and} \quad \nu_2 = \frac{\mu_2}{\mu_0} \, . \tag{33A}$$

(Again we have used $\nu_0 = 1$.) Manipulation of (31) yields the following conclusion:

$$\text{then} \quad b_1 = 0 \quad \text{and} \quad b_2 = 0 \, . \tag{33B}$$

Conversely, imposition of (33B) implies the results in (33A), as may be seen by reference to (31). (The apparent additional solution $\nu_2 = \mu_1^2/\mu_0^2 = \nu_1^2$ would yield an impulse for w and is disallowed.) Thus equality of the first two weighting moments of w with the first two (normalized) moments of probability density function p implies (and is implied by) the vanishing of the first two expansion coefficients b_1 and b_2. This common choice of weighting parameter values can be made if desired, but is not necessary (or recommended) for series approximations by orthonormal polynomials. The equality of probability density function moments in (22) will hold whether (33) is true or not.

EXAMPLE OF DIVERGENT ERROR INTEGRAL FOR $b_1 = 0$, $b_2 = 0$

As a demonstration of what forcing expansion coefficients b_1 and b_2 equal to zero can do, consider probability density function

$$p(u) = \frac{2\, u^{\gamma} \exp(-u^2/\omega^2)}{\omega^{\gamma+1}\, \Gamma\left(\frac{\gamma+1}{2}\right)} \quad \text{for } u > 0 \qquad (\gamma > -1,\ \omega > 0) \tag{34}$$

with moments

$$\mu_n = \omega^n \frac{\Gamma\left(\frac{\gamma+1+n}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} . \tag{35}$$

This class of probability density functions includes the one-sided Gaussian, Rayleigh, and Maxwell as special cases, for $\gamma = 0, 1, 2$, respectively.

Consider also the Hermite weighting given in (1), which has moments (11) equal to

$$\nu_0 = 1, \quad \nu_1 = \alpha, \quad \nu_2 = \alpha^2 + \beta^2 . \tag{36}$$

If we now insist on property (33B), then (33A) yields

$$\alpha = \omega \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} , \qquad \beta^2 = \omega^2 \left[\frac{\gamma+1}{2} - \frac{\Gamma^2\left(\frac{\gamma}{2}+1\right)}{\Gamma^2\left(\frac{\gamma+1}{2}\right)} \right] . \tag{37}$$

But the leading integral in minimum error E_N in (21) is convergent only if $p^2(u)/w(u)$ decays sufficiently rapid for large u. We have from (34) and (1), the dominant behavior

$$p^2(u)/w(u) \propto \exp\left(-\frac{2u^2}{\omega^2} + \frac{u^2}{2\beta^2} \right) \quad \text{for large positive } u , \tag{38}$$

where $\propto$ denotes proportionality, but disregards the exact scale factor and subdominant behavior. Thus the integral in (21) is convergent only if

$$1 < \frac{4\beta^2}{\omega^2} = 2(\gamma+1) - 4\frac{\Gamma^2\left(\frac{\gamma}{2}+1\right)}{\Gamma^2\left(\frac{\gamma+1}{2}\right)} . \tag{39}$$

However, calculation of (39) reveals that this inequality is <u>never</u> satisfied for any value of $\gamma > -1$; the function on the right-hand side starts at 0 when $\gamma = -1$, and increases monotonically towards 1 as $\gamma \to +\infty$, behaving like $1 - 1/(4\gamma)$ in this limit.

Thus expansion of probability density function (34) according to a Hermite weighting has an infinite error integral (21) (and perhaps a divergent series expansion) regardless of the values of γ and ω in the true probability density function, if we insist on expansion coefficients $b_1 = b_2 = 0$. Yet if we relax requirement (33B), and choose β according to (39) such that $\beta > \omega/2$, the error integral in (21) is certainly finite, regardless of α.

However, making the error integral in (21) finite is not the whole story, in so far as realizing useful approximations to the probability density function or cumulative distribution function. An example of probability density function (34) was taken with $\gamma = 3$, $\omega = 1$. When α and β were chosen according to (33) and (37) (giving $\beta = .48 < .5 = \omega/2$), the expansion coefficients $\{b_n\}$ initially decreased to approximately 1E-3 at n = 40 terms, and then diverged; yet a plot of the approximate exceedance distribution function obtained by a Hermite expansion overlaid the exact answer down to the 1E-16 level. On the other hand, when the weighting parameters in the Hermite expansion were chosen as* $\alpha = 0$, $\beta = .7 > .5 = \omega/2$, giving $b_1 \neq 0$ and

*This is example B in a later section

$b_2 \neq 0$, the expansion coefficients $\{b_n\}$ decreased to the 1E-4 level at $n = 70$ before round-off error became dominant; despite this apparent improvement in coefficient level, the approximate exceedance distribution function overlaid a plot of the exact result down to the 1E-10 probability level, which is several orders of magnitude worse than the above result. Thus emphasis on getting a convergent error integral in (21) may not always be desired.

For Hermite weighting (1) and the class of probability density functions which decay as $\exp(-u^q)$ as $u \to +\infty$, the error integral is always convergent if $q > 2$, and always divergent if $q < 2$. So an exponential probability density function, like $u^\gamma \exp(-u/\omega)$ for $u > 0$, always yields a divergent error integral when expanded in a Hermite series.

For generalized Laguerre weighting (2), it is necessary to consider $u = 0^+$ and $u = +\infty$ separately. If probability density function p behaves like u^γ as $u \to 0^+$, then a finite error integral requires that we choose $\alpha < 1 + 2\gamma$. Coupled with the finite area restriction on weighting w, a range of values of α is allowed, namely, $-1 < \alpha < 1 + 2\gamma$; this range always exists since $\gamma > -1$ is necessary for the probability density function itself to have finite area.

If also the probability density function behaves as $\exp(-u/\omega)$ as $u \to +\infty$, then a finite error integral with generalized Laguerre weighting requires that we choose $\beta > \omega/2$. So the range of choice of β is open on the large side, whereas that for α is a limited one, for this particular class of probability density functions.

HERMITE EXPANSION

In this section, we will deal exclusively with weighting (1),

$$w(u) = \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) \quad \text{for all } u \qquad (\beta > 0)\ , \tag{40}$$

where

$$\phi(x) = (2\pi)^{-1/2}\ \exp(-x^2/2), \quad \Phi(x) = \int_{-\infty}^{x} dt\ \phi(t)\ . \tag{41}$$

This weighting has two free parameters, α and β, and moments

$$\nu_0 = 1, \quad \nu_1 = \alpha, \quad \nu_2 = \alpha^2+\beta^2\ . \tag{42}$$

If ν_1 and ν_2 are specified, the parameters must then satisfy $\alpha = \nu_1$, $\beta = (\nu_2 - \nu_1^2)^{1/2}$. However, we shall keep α and β general and unspecified.

PROPERTIES OF POLYNOMIALS AND EXPANSIONS

The orthonormal polynomials associated with weighting (40) are the Hermite polynomials [5, 22.1.2 and 22.2.15]

$$Q_n(u) = He_n\left(\frac{u-\alpha}{\beta}\right)(n!)^{-1/2} \quad \text{for } n \geq 0\ . \tag{43}$$

The expansion coefficients are given by (20) as

$$b_n = \int du\ p(u)\ Q_n(u) = (n!)^{-1/2}\ c_n \quad \text{for } n \geq 0\ , \tag{44}$$

where we define

$$c_n = \int du\ p(u)\ He_n\left(\frac{u-\alpha}{\beta}\right) \quad \text{for } n \geq 0\ . \tag{45}$$

The approximate probability density function then follows from (13) in the form

$$p_N(u) = w(u) \sum_{n=0}^{N} b_n\ Q_n(u) = \frac{1}{\beta}\ \phi\left(\frac{u-\alpha}{\beta}\right) \sum_{n=0}^{N} a_n\ He_n\left(\frac{u-\alpha}{\beta}\right)\ , \tag{46}$$

where we used (40), (43), (44), and defined

$$(n!)^{1/2}\ a_n = b_n = (n!)^{-1/2}\ c_n \quad \text{for } n \geq 0\ . \tag{47}$$

These three different coefficients in (44)-(47) are introduced for convenience in further equation manipulations. Expansion coefficient b_n is the geometric mean of auxiliary coefficients a_n and c_n (with polarity). Expansion (46) is also called a Gram-Charlier series of type A [2, p. 222], [3, p. 156].

The approximate cumulative distribution function corresponding to (46) is

$$P_N(u) = \int_{-\infty}^{u} dt\ p_N(t) = \sum_{n=0}^{N} a_n \int_{-\infty}^{u} \frac{dt}{\beta}\ \phi\left(\frac{t-\alpha}{\beta}\right) He_n\left(\frac{t-\alpha}{\beta}\right) =$$

$$= \sum_{n=0}^{N} a_n \int_{-\infty}^{T} dx\ \phi(x)\ He_n(x) = a_0\ \Phi(T) - \phi(T) \sum_{n=1}^{N} a_n\ He_{n-1}(T)\ , \tag{48}$$

where

$$T = \frac{u-\alpha}{\beta} \tag{49}$$

and we used (41) and [5, 22.11.8].

The Hermite polynomials $\{He_n\}$ satisfy the recurrence [5, 22.7.14]

$$He_n(x) = x\ He_{n-1}(x) - (n-1)\ He_{n-2}(x) \quad \text{for } n \geq 2 \ , \tag{50}$$

with starting values $He_0(x) = 1$, $He_1(x) = x$ [5, 22.3.11]. The highest-order term in $He_n(x)$ is x^n, with coefficient 1 [5, 22.1.2 and 22.3.11]. The magnitude of the term multiplying b_n in (46) has an envelope that decays approximately as $n^{-1/4}$ with n, regardless of argument u. This may be seen by using (47) and (49) to get

$$a_n\ He_n(T) = b_n(n!)^{-1/2}\ He_n(T) \propto b_n\left(n^{n+1/2}\ e^{-n}\right)^{-1/2}\ (n/e)^{n/2} = b_n\ n^{-1/4}$$

$$\text{as } n \to +\infty, \text{ for all } T, \tag{50A}$$

where we also used [5, 6.1.39 and 22.5.18] and [7, 8.22.8]. Here, $\propto$ denotes proportionality and we have taken the magnitude of the terms; the exact scale factor of proportionality will be presented in a later section where the errors of the approximations are estimated. So if b_n were to decay faster than $n^{-3/4}$, the probability density function series in (46) would converge absolutely.

Conditions are better for the cumulative distribution function series in (48); namely, based on the above result, there follows (for the envelope)

$$a_n\ He_{n-1}(T) = b_n(n!)^{-1/2}\ He_{n-1}(T) = b_n\ n^{-1/2}\ [(n-1)!]^{-1/2}\ He_{n-1}(T) =$$

$$\propto b_n\ n^{-1/2}\ n^{-1/4} = b_n\ n^{-3/4} \quad \text{as } n \to +\infty, \text{ for all } T\ . \tag{50B}$$

Thus if b_n decays faster than $n^{-1/4}$, the cumulative distribution function series converges absolutely. Furthermore, if the leading error integral in

(21) is finite, the sum of b_n^2 must be finite, meaning that b_n must decay faster than $n^{-1/2}$. So we can conclude that if the error integral is finite, the Hermite series for the cumulative distribution function in (48) converges. (Notice that this particular decay $n^{-1/2}$ of b_n is not sufficiently fast to make the same conclusion about the Hermite series for the probability density function in (46).) The above are sufficient conditions on expansion coefficients $\{b_n\}$, and are not necessary.

EXPANSION OF CHARACTERISTIC FUNCTION f

The coefficients a_n and c_n were defined in (45) and (47). Then the sum

$$\sum_{n=0}^{\infty} a_n w^n = \sum_{n=0}^{\infty} \frac{1}{n!} c_n w^n = \sum_{n=0}^{\infty} \frac{w^n}{n!} \int du\ p(u)\ He_n\left(\frac{u-\alpha}{\beta}\right) =$$

$$= \int du\ p(u) \sum_{n=0}^{\infty} \frac{w^n}{n!} He_n\left(\frac{u-\alpha}{\beta}\right) = \int du\ p(u) \exp\left(\frac{u-\alpha}{\beta} w - \frac{1}{2} w^2\right) =$$

$$= \exp\left(-\frac{1}{2} w^2 - \frac{\alpha}{\beta} w\right) f\left(\frac{w}{\beta}\right) , \tag{51}$$

where f is the characteristic function, and where we used (45), [5, 22.5.19 and 22.9.17], and (5). Letting $w = \beta z$, we have

$$f(z) \exp\left(-\alpha z - \frac{1}{2} \beta^2 z^2\right) = \sum_{n=0}^{\infty} a_n (\beta z)^n = \sum_{n=0}^{\infty} \frac{1}{n!} c_n (\beta z)^n . \tag{52}$$

Thus $\{a_n\}$ and $\{c_n\}$ are the coefficients in these power series expansions of the function $f(z) \exp(-\alpha z - \beta^2 z^2/2)$, where f is the characteristic function corresponding to probability density function p, and α and β are arbitrary. A special case of (52) is given in [2, 17.6.10].

Collecting (46) and (52) together for comparison, and assuming that $p_n \to p$ as $N \to +\infty$, we have

$$p(u) = \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) \sum_{n=0}^{\infty} a_n \, He_n\left(\frac{u-\alpha}{\beta}\right),$$

$$f(i\xi) = \exp\left(i\alpha\xi - \frac{1}{2}\beta^2\xi^2\right) \sum_{n=0}^{\infty} a_n (i\beta\xi)^n \; . \tag{53}$$

Thus expansion of probability density function p in an infinite Hermite series is equivalent to an expansion of a modified form of the characteristic function in a power series, according to (53). Equations (51)-(53) will serve as very convenient starting points for the derivation of several alternative recurrences for the expansion coefficients $\{a_n\}$. Notice that weighting parameters α and β are completely unrestricted in (52) and (53), except that $\beta > 0$.

An analogous result holds for N finite, but must be derived in a different fashion, because we no longer can use infinite sum [5, 22.9.17]. Define the Fourier transform of (46) as the N-th order approximation to the characteristic function:

$$f_N(i\xi) = \int du \, \exp(i\xi u) \; p_N(u) =$$

$$= \sum_{n=0}^{N} a_n \int du \, \exp(i\xi u) \, \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) He_n\left(\frac{u-\alpha}{\beta}\right) =$$

$$= \sum_{n=0}^{N} a_n \int dt \, \exp(i\xi\alpha + i\xi\beta t) \, \phi(t) \, He_n(t) =$$

$$= \exp(i\alpha\xi) \sum_{n=0}^{N} a_n \int dt \, \exp(i\beta\xi t) \left(-\frac{d}{dt}\right)^n \phi(t) =$$

$$= \exp(i\alpha\xi) \sum_{n=0}^{N} a_n \, (i\beta\xi)^n \int dt \, \exp(i\beta\xi t) \, \phi(t) =$$

$$= \exp\left(i\alpha\xi + \frac{1}{2}\,\beta^2 (i\xi)^2\right) \sum_{n=0}^{N} a_n \, (i\beta\xi)^n \,, \tag{54}$$

where we used [5, 22.11.8] in line 4, and repeated integration by parts in line 5. This result is the leading N terms of (53). As a by-product of this derivation, we have

$$\int dt \, \exp(zt) \, \phi(t) \, He_n(t) = \exp\left(\frac{1}{2}\, z^2\right) z^n \,. \tag{55}$$

COEFFICIENTS RECURSIVELY VIA CUMULANTS

We are now in a position to obtain some useful recursive relations for the expansion coefficients $\{a_n\}$ in (51)-(54). The first one is obtained by taking the ℓn of (51):

$$\ell n \, f\left(\frac{w}{\beta}\right) - \frac{\alpha}{\beta}\, w - \frac{1}{2}\, w^2 = \ell n \left\{\sum_{n=0}^{\infty} a_n \, w^n\right\}. \tag{56}$$

Then using (7) and identifying the right-hand side of (56) as a new power series, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, \chi_n \left(\frac{w}{\beta}\right)^n - \frac{\alpha}{\beta}\, w - \frac{1}{2}\, w^2 = \sum_{n=0}^{\infty} h_n \, w^n \,. \tag{57}$$

There follows immediately

$$h_n = \begin{Bmatrix} \dfrac{\chi_n}{n!\ \beta^n} & \text{for } n \neq 1,2 \\ \dfrac{\chi_1 - \alpha}{\beta} & \text{for } n = 1 \\ \dfrac{1}{2}\left(\dfrac{\chi_2}{\beta^2} - 1\right) & \text{for } n = 2 \end{Bmatrix} . \tag{58}$$

But equality of the right-hand sides of (56) and (57) also requires that

$$\sum_{n=0}^{\infty} a_n\, w^n = \exp\left\{\sum_{n=0}^{\infty} h_n\, w^n\right\}. \tag{59}$$

It is shown in appendix A that a recursive solution to (59) for the $\{a_n\}$ is given by

$$a_n = \frac{1}{n} \sum_{m=1}^{n} m\, h_m\, a_{n-m} \quad \text{for } n \geq 1, \quad a_0 = \exp(h_0) . \tag{60}$$

Then eliminating $\{h_m\}$ by means of (58),

$$a_n = \frac{1}{n}\left[\left(\frac{\chi_1}{\beta} - \frac{\alpha}{\beta}\right) a_{n-1} + \left(\frac{\chi_2}{\beta^2} - 1\right) a_{n-2} + \sum_{m=3}^{n} \frac{\chi_m}{(m-1)!\ \beta^m}\, a_{n-m}\right] \quad \text{for } n \geq 1,$$

$$a_0 = \exp(\chi_0), \tag{61}$$

where $a_n \equiv 0$ for $n < 0$, and the sum is zero for $n < 3$.

Now define normalized cumulants (excluding n=0) according to

$$\hat{\chi}_n = \frac{\chi_n}{(n-1)!\ \beta^n} \quad \text{for } n \geq 1 . \tag{62}$$

Then (61) becomes

$$a_n = \frac{1}{n}\left[\left(\hat{\chi}_1 - \frac{\alpha}{\beta}\right) a_{n-1} + \left(\hat{\chi}_2 - 1\right) a_{n-2} + \sum_{m=3}^{n} \hat{\chi}_m \, a_{n-m}\right] \quad \text{for } n \geq 1 \, ,$$

$$a_0 = \exp(\chi_0) \, . \qquad (63)$$

This convolution is the desired recursion for expansion coefficients $\{a_n\}$ via cumulants.

As particular cases, we have

$$a_1 = \frac{\chi_1 - \alpha}{\beta} a_0, \quad a_2 = \frac{1}{2}\left[\left(\frac{\chi_1 - \alpha}{\beta}\right)^2 + \frac{\chi_2}{\beta^2} - 1\right] a_0 \, . \qquad (64A)$$

Parameters α and β (>0) are completely arbitrary in the above three equations, and $\{\chi_n\}_0^N$ are the available cumulants of the probability density function under consideration.

Observe that if we choose $\alpha = \chi_1 = M$ and $\beta = \chi_2^{1/2} = R$ (see (24)-(25)), which is a very common choice, we have $a_1 = 0$ and $a_2 = 0$; this is a special case of the general property (33) stated earlier. This special choice of α and β corresponds to choosing the mean location and rms width of Hermite weighting (40) identical to those same parameters of the given probability density function. There then also follows, in this special case,

$$a_3 = \frac{1}{3}\hat{\chi}_3 \, a_0 \, , \quad a_4 = \frac{1}{4}\hat{\chi}_4 \, a_0 \, , \quad a_5 = \frac{1}{5}\hat{\chi}_5 a_0 \, ,$$

$$a_n = \frac{1}{n}\left[\hat{\chi}_n \, a_0 + \sum_{m=3}^{n-3} \hat{\chi}_m \, a_{n-m}\right] \quad \text{for } n \geq 6 \quad \text{when } \alpha = \chi_1, \ \beta = \chi_2^{1/2} \, . \qquad (64B)$$

COEFFICIENTS DIRECTLY VIA MOMENTS

Before we begin this derivation, we present the following useful expansion [5, 22.9.17 and 22.5.19]:

$$\exp\left(-\frac{1}{2}y^2 + xy\right) = \sum_{n=0}^{\infty} \frac{1}{n!} He_n(x)\, y^n . \tag{65}$$

We now again refer to (51) and expand the terms as follows:

$$\sum_{n=0}^{\infty} a_n w^n = \exp\left(-\frac{1}{2}w^2 - \frac{\alpha}{\beta}w\right) f\left(\frac{w}{\beta}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} He_k\left(-\frac{\alpha}{\beta}\right) w^k \sum_{m=0}^{\infty} \frac{1}{m!} \mu_m \left(\frac{w}{\beta}\right)^m , \tag{66}$$

where we utilized (65) and (6). Equating coefficients of w^n on both sides of this equation, we have

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} He_k\left(-\frac{\alpha}{\beta}\right) \frac{\mu_{n-k}}{(n-k)!\, \beta^{n-k}} \quad \text{for } n \geq 0 . \tag{67}$$

We now define, for convenience, the normalized Hermite polynomials

$$\hat{He}_n(x) = \frac{1}{n!} He_n(x) \quad \text{for } n \geq 0 , \tag{68}$$

and the normalized moments

$$\hat{\mu}_n = \frac{\mu_n}{n!\, \beta^n} \quad \text{for } n \geq 0 . \tag{69}$$

(Notice the difference with the definition of the normalized cumulants (62).) Then (67) becomes

$$a_n = \sum_{k=0}^{n} \hat{He}_k\left(-\frac{\alpha}{\beta}\right) \hat{\mu}_{n-k} \quad \text{for } n \geq 0 \,, \tag{70}$$

which gives expansion coefficients $\{a_n\}$ directly in terms of the (normalized) moments of the given probability density function. The recurrence in (50) can be used to generate the Hermite factors needed in convolution (70). Parameters α and β (>0) of weighting (40) are arbitrary.

$\hat{He}_0(x) = 1, \quad \hat{He}_1(x) = x, \quad \hat{He}_n(x) = \frac{1}{n}\left[x\, \hat{He}_{n-1}(x) - \hat{He}_{n-2}(x)\right]$ for $n \geq 2$.

As particular cases, we have

$$a_0 = \mu_0, \quad a_1 = \frac{\mu_1 - \alpha\mu_0}{\beta}, \quad a_2 = \frac{\mu_2 - 2\alpha\mu_1 + (\alpha^2-\beta^2)\,\mu_0}{2\beta^2} \,. \tag{71}$$

These agree with (64) which utilized cumulants. If we make the special choice of $\alpha = \mu_1/\mu_0$ and $\beta^2 = \mu_2/\mu_0 - (\mu_1/\mu_0)^2$, then $a_1 = 0$ and $a_2 = 0$.

An alternative more direct derivation of (67) is possible: from (47), (45), (B-3) in appendix B, and (3),

$$a_n = \frac{1}{n!} c_n = \frac{1}{n!} \int du\; p(u)\; He_n\left(\frac{u-\alpha}{\beta}\right) =$$

$$= \frac{1}{n!} \int du\; p(u) \sum_{k=0}^{n} \binom{n}{k} He_k\left(-\frac{\alpha}{\beta}\right)\left(\frac{u}{\beta}\right)^{n-k} =$$

$$= \sum_{k=0}^{n} \frac{1}{k!} He_k\left(-\frac{\alpha}{\beta}\right) \frac{\mu_{n-k}}{(n-k)!\; \beta^{n-k}} \quad \text{for } n \geq 0 \,. \tag{72}$$

COEFFICIENTS RECURSIVELY VIA MOMENTS

Before we begin this derivation, we replace $x \to -ix$, $y \to iy$ in (65) to get

$$\exp\left(\frac{1}{2} y^2 + xy\right) = \sum_{n=0}^{\infty} \frac{1}{n!} He_n(-ix)\,(iy)^n = \sum_{n=0}^{\infty} \frac{1}{n!} Hi_n(x)\, y^n \,, \tag{73}$$

where $Hi_n(x)$ is a real n-th order modified Hermite polynomial in x defined by

$$Hi_n(x) = i^n\, He_n(-ix) \quad \text{for } n \geq 0 \,. \tag{74}$$

The recursion for these polynomials follows immediately from (50) as

$$Hi_n(x) = x\, Hi_{n-1}(x) + (n-1)\, Hi_{n-2}(x) \quad \text{for } n \geq 2 \,, \tag{75}$$

with starting values $Hi_0(x) = 1$, $Hi_1(x) = x$. The difference with (50) is the polarity of the last term; thus for example, $Hi_2(x) = x^2 + 1$, $Hi_3(x) = x^3 + 3x$, versus $He_2(x) = x^2 - 1$, $He_3(x) = x^3 - 3x$.

We now rewrite (51) in the following form:

$$f\left(\frac{w}{\beta}\right) = \exp\left(\frac{1}{2} w^2 + \frac{\alpha}{\beta} w\right) \sum_{m=0}^{\infty} a_m\, w^m \,. \tag{76}$$

Expanding in power series by means of (6) and (73),

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mu_n \left(\frac{w}{\beta}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!} Hi_k\left(\frac{\alpha}{\beta}\right) w^k \sum_{m=0}^{\infty} a_m\, w^m \,. \tag{77}$$

Equating coefficients of w^n, there follows

$$\frac{\mu_n}{n!\, \beta^n} = \sum_{k=0}^{n} \frac{1}{k!} Hi_k\left(\frac{\alpha}{\beta}\right) a_{n-k} \quad \text{for } n \geq 0 \,, \tag{78}$$

or

$$\hat{\mu}_n = \sum_{k=0}^{n} \hat{Hi}_k\left(\frac{\alpha}{\beta}\right) a_{n-k} = a_n + \sum_{k=1}^{n} \hat{Hi}_k\left(\frac{\alpha}{\beta}\right) a_{n-k} \quad \text{for } n \geq 0 \ , \tag{79}$$

where we have used normalized moments (69), and defined the normalized modified Hermite polynomials

$$\hat{Hi}_n(x) = \frac{1}{n!} Hi_n(x) \quad \text{for } n \geq 0 \ . \tag{80}$$

Finally, the desired recursion for expansion coefficients $\{a_n\}$ in terms of the moments follows as

$$a_n = \hat{\mu}_n - \sum_{k=1}^{n} \hat{Hi}_k\left(\frac{\alpha}{\beta}\right) a_{n-k} \quad \text{for } n \geq 0 \ . \tag{81}$$

Parameters α and β (>0) are arbitrary in (81) and (69).

SUMMARY

The approximations to the probability density function and cumulative distribution function are given by (46) and (48), respectively, where α and β are arbitrary constants, except that $\beta > 0$. The functions ϕ and Φ are defined in (41), while the Hermite polynomials $\{He_n\}$ are available via (50). The expansion coefficients $\{a_n\}$ are given by the three alternatives (63), (70), (81), in terms of normalized cumulants (62), normalized moments (69), normalized Hermite polynomials (68), and normalized modified Hermite polynomials (80) and (74). Programs for all three alternative procedures for determining expansion coefficients $\{a_n\}$ are listed in an appendix. The basis for these relations is the characteristics function expansion in (51)-(53).

GENERALIZED LAGUERRE EXPANSION

This section will treat weighting (2), namely,

$$w(u) = \frac{u^{\alpha} \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \quad \text{for } u > 0 \qquad (\alpha > -1,\ \beta > 0). \tag{82}$$

This weighting is a special case of the three-parameter weighting

$$\frac{(u-\gamma)^{\alpha} \exp\left(-\frac{u-\gamma}{\beta}\right)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \quad \text{for } u > \gamma\ , \tag{83}$$

which is the most general scaled linear shift of the generalized Laguerre weighting [5, 22.2.12]

$$x^{\alpha} \exp(-x) \quad \text{for } x > 0\ . \tag{84}$$

We will consider only $\gamma = 0$ here. For a probability density function $p_0(u)$ which is known to be nonzero only for $u > u_0$, we would consider the modified probability density function $p(u) = p_0(u+u_0)$, because then $p(u) \neq 0$ only for $u > 0$, and the simpler weighting (82) would be directly applicable. This procedure is equivalent to choosing $\gamma = u_0$ in the three-parameter weighting (83) above, and requires knowledge of u_0. We presume that $p(u) \neq 0$ only for $u > 0$ henceforth in this section, and that any necessary shifting has already taken place.

Weighting (82) has two free parameters, α and β, and moments

$$\nu_n = (\alpha+1)_n\, \beta^n \quad \text{for } n \geq 0\ . \tag{85}$$

In particular,

$$\nu_0 = 1, \quad \nu_1 = (\alpha+1)\beta, \quad \nu_2 = (\alpha+2)(\alpha+1)\beta^2 . \tag{86}$$

If ν_1 and ν_2 are specified, then the parameters must satisfy

$$\alpha = \frac{\nu_1^2}{\nu_2 - \nu_1^2} - 1 , \quad \beta = \frac{\nu_2 - \nu_1^2}{\nu_1} . \tag{87}$$

However, we shall keep α and β general and unspecified except for the conditions in (82).

PROPERTIES OF POLYNOMIALS AND EXPANSIONS

The orthonormal polynomials associated with weighting (82) are the generalized Laguerre polynomials [5, 22.1.2 and 22.2.12]

$$Q_n(u) = L_n^{(\alpha)}\left(\frac{u}{\beta}\right)\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} \quad \text{for } n \geq 0, \quad u > 0 . \tag{88}$$

The expansion coefficients are given by (20) as

$$b_n = \int_0^\infty du\ p(u)\ Q_n(u) = \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} c_n \quad \text{for } n \geq 0 , \tag{89}$$

where we define

$$c_n = \int_0^\infty du\ p(u)\ L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \quad \text{for } n \geq 0 . \tag{90}$$

The approximate probability density function follows from (13) according to

$$p_N(u) = w(u) \sum_{n=0}^{N} b_n\, Q_n(u) =$$

$$= \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n\, L_n^{(\alpha)}\!\left(\frac{u}{\beta}\right) \quad \text{for } u > 0\ , \tag{91}$$

where we used (82), (88), (89), and defined

$$\left(\frac{(\alpha+1)_n}{n!}\right)^{1/2} a_n = b_n = \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} c_n \quad \text{for } n \geq 0\ . \tag{92}$$

These three different coefficients in (89)-(92) are introduced for convenience in further equation manipulations. Expansion coefficient b_n is the geometric mean of auxiliary coefficients a_n and c_n (with polarity).

The approximate cumulative distribution function corresponding to (91) is

$$P_N(u) = \int_0^u dt\; p_N(t) = \sum_{n=0}^{N} a_n \int_0^u dt\, \frac{t^\alpha \exp(-t/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)}\, L_n^{(\alpha)}\!\left(\frac{t}{\beta}\right) =$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n\, I_n\!\left(\frac{u}{\beta}\right) \quad \text{for } u > 0\ , \tag{93}$$

where we define

$$I_n(y) = \int_0^y dx\; x^\alpha e^{-x} L_n^{(\alpha)}(x) \quad \text{for } n \geq 0,\quad y > 0\ . \tag{94}$$

These quantities are evaluated in appendix C; when substituted in (93), they yield (95)

$$P_N(u) = \frac{(u/\beta)^{\alpha+1} \exp(-u/\beta)}{\Gamma(\alpha+1)} \left[\frac{a_0}{\alpha+1}\, {}_1F_1(1;\alpha+2;\tfrac{u}{\beta}) + \sum_{n=1}^{N} \frac{a_n}{n}\, L_{n-1}^{(\alpha+1)}\!\left(\frac{u}{\beta}\right)\right] \quad \text{for } u > 0,$$

where ${}_1F_1$ is the confluent hypergeometric function.

The generalized Laguerre polynomials $\left\{L_n^{(\alpha)}\right\}$ satisfy the recurrence [5, 22.7.12]

$$L_n^{(\alpha)}(x) = \frac{1}{n}\left[(\alpha-1+2n-x)\ L_{n-1}^{(\alpha)}(x) - (\alpha-1+n)\ L_{n-2}^{(\alpha)}(x)\right] \quad \text{for } n \geq 2 , \tag{96}$$

with starting values $L_0^{(\alpha)}(x) = 1$, $L_1^{(\alpha)}(x) = \alpha+1-x$ [5, 22.4.7]. The highest order term in $L_n^{(\alpha)}(x)$ is $(-x)^n/n!$ [5, 22.1.2 and 22.3.9]; this is distinctly different from the coefficient 1 for the Hermite polynomials. Yet the envelope decay with n of the generalized Laguerre series for the probability density function and cumulative distribution function are identical to those of the Hermite series, for $u > 0$. To prove this, use (91) and (92) to get

$$a_n\ L_n^{(\alpha)}\left(\frac{u}{\beta}\right) = b_n\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \propto b_n\left(n^{-\alpha}\right)^{\frac{1}{2}}\ n^{\frac{\alpha}{2}-\frac{1}{4}} = b_n\ n^{-\frac{1}{4}}$$

$$\text{as } n \to +\infty, \quad \text{for } u > 0 , \tag{97}$$

where we also used [5, 6.1.39] and [7, 8.22.1]. Again, $\propto$ denotes proportionality with n only; the exact scale factor will be presented in a later section where the errors of the approximations are estimated. So if b_n decays faster than $n^{-3/4}$, the probability density function series in (91) converges absolutely.

For the generalized Laguerre series of the cumulative distribution function in (95), we have, for the envelope of the general term,

$$\frac{1}{n} a_n L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right) = b_n \frac{1}{n} \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right) =$$

$$\propto b_n \frac{1}{n} \left(n^{-\alpha}\right)^{\frac{1}{2}} (n-1)^{\frac{\alpha+1}{2} - \frac{1}{4}} \sim b_n n^{-3/4} \text{ as } n \to +\infty, \quad \text{for } u > 0 . \tag{98}$$

Thus if b_n decays faster than $n^{-1/4}$, (95) converges absolutely. And if the error integral (21) is finite, this property of the $\{b_n\}$ is true. So if error integral (21) is finite, the generalized Laguerre series for the cumulative distribution function converges absolutely for $u > 0$; this is a sufficient, but not necessary, condition.

For zero argument, the generalized Laguerre polynomials behave differently for large n. From [5, 22.4.7 and 6.1.39],

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!} \sim \frac{n^\alpha}{\Gamma(\alpha+1)} \text{ as } n \to +\infty. \tag{99}$$

Then (97) and (98) are both replaced by $b_n n^{\alpha/2}$ as $n \to +\infty$. However, for $\alpha > 0$, the probability density function in (91) is zero at $u = 0$ due to the u^α term, so there is no need to perform the sum then. And the cumulative distribution function is always zero at $u = 0$, again eliminating the need to evaluate the sum in (95). So the difference in behavior at $u = 0$ is of no consequence.

EXPANSION OF CHARACTERISTIC FUNCTION f

The coefficients a_n and c_n for the generalized Laguerre series were defined in (90) and (92). Then the sum

$$\sum_{n=0}^{\infty} c_n w^n = \sum_{n=0}^{\infty} w^n \int_0^{\infty} du\ p(u)\ L_n^{(\alpha)}(u/\beta) =$$

$$= \int_0^{\infty} du\ p(u) \sum_{n=0}^{\infty} w^n L_n^{(\alpha)}(u/\beta) = \int_0^{\infty} du\ p(u)\ (1-w)^{-\alpha-1} \exp\left(-\frac{uw/\beta}{1-w}\right) =$$

$$= (1-w)^{-\alpha-1} f\left(\frac{-w/\beta}{1-w}\right), \tag{100}$$

where f is the characteristic function, and where we used (90), [5, 22.9.15], and (5). Thus $\{c_n\}$ are the expansion coefficients of the right-hand side of (100) in powers of w. If we let $w = \frac{-\beta z}{1-\beta z}$, we have the expansion for the characteristic function

$$f(z) = (1-\beta z)^{-\alpha-1} \sum_{n=0}^{\infty} c_n \left(\frac{-\beta z}{1-\beta z}\right)^n, \tag{101}$$

corresponding to given probability density function p. Weighting parameters α and β are arbitrary in (100) and (101).

Collecting (91) and (101) together for comparison, and assuming that $p_N \to p$ as $N \to +\infty$, we have, upon use of (92),

$$p(u) = \frac{u^{\alpha} \exp(-u/\beta)}{\beta^{\alpha+1}\, \Gamma(\alpha+1)} \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \quad \text{for } u > 0,$$

$$f(i\xi) = (1-i\beta\xi)^{-\alpha-1} \sum_{n=0}^{\infty} a_n \frac{(\alpha+1)_n}{n!} \left(\frac{-i\beta\xi}{1-i\beta\xi}\right)^n. \tag{102}$$

Thus, expansion of probability density function p in an infinite generalized Laguerre series is equivalent to an expansion of the corresponding characteristic function in the series of the particular form in (102). Equations (100)-(102) will serve as very convenient starting points for the derivation of several alternative recurrences for expansion coefficients $\{a_n\}$. We reiterate that α and β are arbitrary in the above, except that $\alpha > -1$, $\beta > 0$.

An analogous result holds for N finite, but must be derived differently since we can no longer use infinite sum [5, 22.9.15]. Define the Fourier transform of (91) as the N-th order approximation to the characteristic function:

$$f_N(i\xi) = \int du \, \exp(i\xi u) \, p_N(u) =$$

$$= \int_0^\infty du \, \exp(i\xi u) \, \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1} \, \Gamma(\alpha+1)} \sum_{n=0}^{N} a_n \, L_n^{(\alpha)}\left(\frac{u}{\beta}\right) =$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n \int_0^\infty dt \, \exp(i\beta\xi t) \, t^\alpha \, e^{-t} \, L_n^{(\alpha)}(t) \, . \tag{103}$$

In appendix D, it is shown that

$$\int_0^\infty dt \, e^{i\omega t} \, t^\alpha \, e^{-t} \, L_n^{(\alpha)}(t) = \frac{\Gamma(\alpha+1+n)}{n!} \, \frac{(-i\omega)^n}{(1-i\omega)^{\alpha+1+n}} \, . \tag{104}$$

Substitution in (103) then yields

$$f_N(i\xi) = \sum_{n=0}^{N} a_n \frac{(\alpha+1)_n}{n!} \, \frac{(-i\beta\xi)^n}{(1-i\beta\xi)^{\alpha+1+n}} = \sum_{n=0}^{N} c_n \, \frac{(-i\beta\xi)^n}{(1-i\beta\xi)^{\alpha+1+n}} \, , \tag{105}$$

where the last relation follows by use of (92). This result is the leading N terms of (102).

COEFFICIENTS RECURSIVELY VIA CUMULANTS

We can now obtain some useful recursive relations for expansion coefficients $\{a_n\}$ and/or $\{c_n\}$ in (100)-(105). We start by taking the ℓn of (100):

$$\ell n \left\{ \sum_{n=0}^{\infty} c_n \, w^n \right\} = -(\alpha+1) \, \ell n(1-w) + \ell n \, f\left(\frac{-w/\beta}{1-w}\right) . \tag{106}$$

Identify the left-hand side as a new power series, and use (7) and [5, 15.1.8] to yield

$$\sum_{n=0}^{\infty} h_n \, w^n = (\alpha+1) \sum_{n=1}^{\infty} \frac{1}{n} w^n + \sum_{k=0}^{\infty} \frac{1}{k!} \chi_k \left(\frac{-w/\beta}{1-w}\right)^k =$$

$$= (\alpha+1) \sum_{n=1}^{\infty} \frac{1}{n} w^n + \sum_{k=0}^{\infty} \frac{(-1)^k \chi_k}{k! \, \beta^k} w^k \sum_{m=0}^{\infty} \frac{(k)_m}{m!} w^m . \tag{107}$$

Equating coefficients of w^n, there follows $h_0 = \chi_0$, while for $n \geq 1$,

$$h_n = \frac{1}{n}(\alpha+1) + \sum_{k=0}^{n} \frac{(-1)^k \chi_k}{k! \, \beta^k} \, \frac{(k)_{n-k}}{(n-k)!} =$$

$$= \frac{1}{n} \left[\alpha+1+ \sum_{k=1}^{n} (-1)^k \binom{n}{k} \hat{\chi}_k \right] , \tag{108}$$

where we used the normalized cumulants defined in (62).

But since the left-hand sides of (106) and (107) are equal, we have

$$\sum_{n=0}^{\infty} c_n \, w^n = \exp\left\{ \sum_{n=0}^{\infty} h_n \, w^n \right\} , \tag{109}$$

or via appendix A, the recurrence

$$c_n = \frac{1}{n} \sum_{m=1}^{n} m\, h_m\, c_{n-m} \quad \text{for } n \geq 1, \quad c_o = \exp(h_o) \ . \tag{110}$$

Finally, define

$$d_m = m\, h_m \quad \text{for } m \geq 1 \tag{111}$$

for notational convenience and thereby obtain

$$d_m = \alpha + 1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \hat{\chi}_k \quad \text{for } m \geq 1 \ ,$$

$$c_n = \frac{1}{n} \sum_{m=1}^{n} d_m\, c_{n-m} \quad \text{for } n \geq 1, \quad c_0 = \exp(\chi_0) \ , \tag{112}$$

by means of (108) and (110), respectively. Equation (112) is a recursive relation for expansion coefficients $\{c_n\}$ in terms of cumulants $\{\chi_n\}$ and auxiliary variables $\{d_m\}$. The $\{a_n\}$ are immediately available via (92).

As particular cases, we have, employing (62),

$$c_1 = \left(\alpha + 1 - \frac{\chi_1}{\beta}\right) c_0 \ ,$$

$$c_2 = \frac{1}{2} \left[(\alpha+2)(\alpha+1) - 2(\alpha+2)\frac{\chi_1}{\beta} + \frac{\chi_2 + \chi_1^2}{\beta^2} \right] c_0 \ . \tag{113}$$

Parameters α and β are completely arbitrary in all the above equations, except that $\alpha > -1$ and $\beta > 0$, and $\{\chi_n\}_0^N$ are the available cumulants.

Observe that if we

$$\text{let} \qquad \alpha + 1 = \frac{\chi_1^2}{\chi_2} = \frac{\mu_1^2}{\mu_2\mu_0 - \mu_1^2} = \frac{M^2}{R^2}$$

$$\text{and} \qquad \beta = \frac{\chi_2}{\chi_1} = \frac{\mu_2\mu_0 - \mu_1^2}{\mu_1\mu_0} = \frac{R^2}{M} \ , \tag{114}$$

then $c_1 = 0$ and $c_2 = 0$ (here we also used (8) and (25)); this is a special case of general property (33) stated earlier. Since the probability density function p(u) is nonzero only for u > 0, then $\chi_1 > 0$ and $\chi_2 > 0$, giving allowable solutions to (114) in all cases. There then also follows, along with $d_1 = d_2 = 0$ in this special case, the explicit results

$$
\begin{aligned}
c_0 &= \exp(\chi_0), \quad c_1 = 0, \quad c_2 = 0, \\
c_3 &= \frac{\chi_1^2}{3!\,\chi_2^3}\,(2\chi_2^2 - \chi_3\chi_1)\, c_0, \\
c_4 &= \frac{\chi_1^2}{4!\,\chi_2^4}\,(18\chi_2^3 - 12\chi_3\chi_2\chi_1 + \chi_4\chi_1^2)\, c_0, \\
c_5 &= \frac{\chi_1^2}{5!\,\chi_2^5}\,(144\chi_2^4 - 120\chi_3\chi_2^2\chi_1 + 20\chi_4\chi_2\chi_1^2 - \chi_5\chi_1^3)\, c_0, \\
c_6 &= \frac{\chi_1^2}{6!\,\chi_2^6}\Big(40(30\chi_2 + \chi_1^2)(\chi_2^2 - \chi_3\chi_1)\chi_2^2 + 10\chi_3^2\chi_1^4 + \\
&\qquad + 300\chi_4\chi_2^2\chi_1^2 - 30\chi_5\chi_2\chi_1^3 + \chi_6\chi_1^4\Big)\, c_0, \\
c_7 &= \frac{\chi_1^2}{7!\,\chi_2^7}\Big(10800\chi_2^6 + 1260\chi_2^5\chi_1^2 - 12600\chi_3\chi_2^4\chi_1 - 1470\chi_3\chi_2^3\chi_1^3 + \\
&\quad + 420\chi_3^2\chi_2\chi_1^4 + 4200\chi_4\chi_2^3\chi_1^2 + 70\chi_4\chi_2^2\chi_1^4 - 35\chi_4\chi_3\chi_1^5 - \\
&\qquad - 630\chi_5\chi_2^2\chi_1^3 + 42\chi_6\chi_2\chi_1^4 - \chi_7\chi_1^5\Big)\, c_0.
\end{aligned}
\tag{115}
$$

These relations have been confirmed by numerical comparison with (112).

These results greatly extend those of [1, (129)-(131)], where the equivalent of our c_3 is given (in terms of moments instead of cumulants, and with $\chi_0 = 0$), along with the comment that "the higher-order coefficients are so complicated that the whole value of this type of series seems to depend on the fact that the first term alone (c_0) is often a good approximation." We find, on the other hand, that not only can we avoid the special choice in (114) and the corresponding complicated special results in (115), but we can handle any α,β pair and get very high-order coefficients c_n, simply by using the recurrence in (112), which is easily programmed. The only thing we lose are explicit results of the type given in (115); however, the latter are so complicated that they are of limited utility anyway.

COEFFICIENTS DIRECTLY VIA MOMENTS

We will need the following expression [5, 22.3.9]:

$$\frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x) = \frac{n!}{(\alpha+1)_n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{(\alpha+1)_k} . \tag{116}$$

Then (92), (90), and (3) yield, for $n \geq 0$,

$$a_n = \frac{n!}{(\alpha+1)_n} c_n = \frac{n!}{(\alpha+1)_n} \int_0^\infty du\ p(u)\ L_n^{(\alpha)}\left(\frac{u}{\beta}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(\alpha+1)_k} \int_0^\infty du\ p(u)\ (-u/\beta)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\mu_k}{(\alpha+1)_k\ \beta^k} . \tag{117}$$

It is useful, in this generalized Laguerre series case, to define an alternative set of normalized moments

$$\tilde{\mu}_n = \frac{\mu_n}{(\alpha+1)_n \, \beta^n} \quad \text{for } n \geq 0 \; . \tag{118}$$

(Although this seems to be very different from the earlier normalization in (69), (118) actually reduces to (69) for the α here equal to zero.) When (118) is utilized in (117), we have the desired expression for expansion coefficients $\{a_n\}$, directly in terms of (normalized) moments, in the surprisingly simple form

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \tilde{\mu}_k \quad \text{for } n \geq 0 \; . \tag{119}$$

Parameters α and β in (118) are arbitrary, except that $\alpha > -1$, $\beta > 0$.

As particular cases, (117)-(119) yield

$$a_0 = \mu_0, \quad a_1 = \mu_0 - \frac{\mu_1}{(\alpha+1)\beta}, \quad a_2 = \mu_0 - \frac{2\mu_1}{(\alpha+1)\beta} + \frac{\mu_2}{(\alpha+1)(\alpha+2)\beta^2} \; . \tag{120}$$

These agree with (113) which utilized cumulants. If we make the special choices of $\alpha+1 = \mu_1^2/(\mu_2\mu_0 - \mu_1^2)$ and $\beta = (\mu_2\mu_0 - \mu_1^2)/(\mu_1\mu_0)$, then $a_1 = 0$ and $a_2 = 0$; this is a common approach to the approximation problem, but totally unnecessary.

An alternative derivation of the direct moment relation (117) is possible: from (100), (6), and [5, 15.1.8],

$$\sum_{n=0}^{\infty} c_n \, w^n = (1-w)^{-\alpha-1} \sum_{k=0}^{\infty} \frac{1}{k!} \, \mu_k \left(\frac{-w/\beta}{1-w}\right)^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \, \mu_k \left(-\frac{w}{\beta}\right)^k (1-w)^{-\alpha-1-k} = \sum_{k=0}^{\infty} \frac{1}{k!} \, \mu_k \left(-\frac{w}{\beta}\right)^k \sum_{m=0}^{\infty} \frac{(\alpha+1+k)_m}{m!} \, w^m \, . \qquad (121)$$

Equating coefficients of w^n, we have, for $n \geq 0$,

$$c_n = \sum_{k=0}^{n} \frac{1}{k!} \, \mu_k \left(-\frac{1}{\beta}\right)^k \frac{(\alpha+1+k)_{n-k}}{(n-k)!} = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\mu_k}{(\alpha+1)_k \, \beta^k} \, , \qquad (122)$$

which is equivalent to (117).

COEFFICIENTS RECURSIVELY VIA MOMENTS

The starting point for this case is the characteristic function expansion in (101):

$$f(z) = \sum_{m=0}^{\infty} c_m \, (-\beta z)^m \, (1-\beta z)^{-\alpha-1-m} = \sum_{m=0}^{\infty} c_m \, (-\beta z)^m \sum_{k=0}^{\infty} \frac{(\alpha+1+m)_k}{k!} \, (\beta z)^k \qquad (123)$$

by use of [5, 15.1.8]. Now expand the left-hand side of (123) in powers of z, according to (6), and equate the coefficients of z^n to get, for $n \geq 0$,

$$\frac{1}{n!} \, \mu_n = \sum_{m=0}^{n} c_m \, (-\beta)^m \frac{(\alpha+1+m)_{n-m}}{(n-m)!} \, \beta^{n-m} = \beta^n \sum_{m=0}^{n} \frac{c_m (-1)^m}{(n-m)!} \frac{(\alpha+1)_n}{(\alpha+1)_m} \, . \qquad (124)$$

Therefore

$$\frac{\mu_n}{(\alpha+1)_n \, \beta^n} = \sum_{m=0}^{n} (-1)^m \frac{n!}{(n-m)!} \frac{c_m}{(\alpha+1)_m} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} a_m \quad \text{for } n \geq 0 \, , \qquad (125)$$

by use of (92). Then using normalized moment definition (118), (125) can be expressed as

$$a_n = (-1)^n \left[\tilde{\mu}_n - \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} a_m \right] \quad \text{for } n \geq 0 . \tag{126}$$

This is a recursive relation for expansion coefficients $\{a_n\}$ in terms of (normalized) moments. The parameters α and β in (118) are arbitrary, except that $\alpha > -1$, $\beta > 0$.

SUMMARY

The approximations to the probability density function and cumulative distribution function are given by (91) and (95), respectively, where α and β are arbitrary constants, except that $\alpha > -1$, $\beta > 0$. The generalized Laguerre polynomials are available via (96). The expansion coefficients $\{a_n\}$ are given by the three alternatives (112), (119), (126), in terms of normalized cumulants (62) and normalized moments (118); in the case of (112), the interrelationship between expansion coefficients $\{a_n\}$ and $\{c_n\}$ is given in (92). Programs for all three alternative procedures for determining the expansion coefficients $\{a_n\}$ are listed in an appendix. The basis for these relations is the characteristic function expansion in (100)-(102).

EXAMPLES OF HERMITE EXPANSION

EXAMPLE A

The first example is one which can be handled analytically, and thereby furnishes checks on numerical procedures and results. Consider the Gaussian probability density function

$$p(u) = \frac{1}{\omega}\,\phi\left(\frac{u-\gamma}{\omega}\right) \qquad (\omega > 0) \tag{127A}$$

with cumulative distribution function and characteristic function

$$P(u) = \Phi\left(\frac{u-\gamma}{\omega}\right), \quad f(i\xi) = \exp\left(i\xi\gamma - \frac{1}{2}\xi^2\omega^2\right). \tag{127B}$$

The cumulants are

$$\chi_0 = 0, \quad \chi_1 = \gamma, \quad \chi_2 = \omega^2, \quad \chi_n = 0 \quad \text{for } n \geq 3, \tag{128A}$$

while the moments are most easily evaluated by the recurrence

$$\mu_n = \gamma\,\mu_{n-1} + (n-1)\,\omega^2\,\mu_{n-2} \quad \text{for } n \geq 2, \quad \mu_0 = 1, \quad \mu_1 = \gamma. \tag{128B}$$

It is obvious in this Hermite expansion case that the best choice of weighting parameters would be $\alpha = \gamma$, $\beta = \omega$, for then weighting w would match p perfectly and there would follow $b_n = 0$ for $n \geq 1$. We consider a mismatched choice of α and β to illustrate rapid decay of the expansion coefficients and some conditions on convergence.

Expansion coefficient c_n follows from (45) and (127A) according to

$$c_n = \int du\ p(u)\ He_n\left(\frac{u-\alpha}{\beta}\right) = \beta \int dx\ p(\alpha+\beta x)\ He_n(x) =$$

$$= \beta \int dx\ \frac{1}{\omega}\ \phi\left(\frac{\alpha-\gamma+\beta x}{\omega}\right)\ He_n(x) = \left(\frac{\sqrt{\beta^2-\omega^2}}{\beta}\right)^n\ He_n\left(\frac{\gamma-\alpha}{\sqrt{\beta^2-\omega^2}}\right), \tag{129}$$

the last step via use of [5, 22.5.18] and [8, 7.374 10]. Then from (47),

$$b_n = (n!)^{-1/2} \left(\frac{\sqrt{\beta^2-\omega^2}}{\beta}\right)^n\ He_n\left(\frac{\gamma-\alpha}{\sqrt{\beta^2-\omega^2}}\right) \quad \text{for } n \geq 0\ . \tag{130}$$

This equation is correct for all positive values of β and ω. However, for $\beta < \omega$, a more convenient form can be obtained by use of (74), if desired:

$$b_n = (n!)^{-1/2} \left(\frac{\sqrt{\omega^2-\beta^2}}{\beta}\right)^n\ Hi_n\left(\frac{\gamma-\alpha}{\sqrt{\omega^2-\beta^2}}\right), \tag{131}$$

where Hi_n is the modified Hermite polynomial. For $\beta = \omega$, a limit of (130) yields $b_n = (n!)^{-1/2}((\gamma-\alpha)/\beta)^n$.

If $\beta > \omega$, we can use the result in (50A) on (130) and obtain

$$b_n \propto \left(\frac{\sqrt{\beta^2-\omega^2}}{\beta}\right)^n\ n^{-1/4} \quad \text{as } n \to +\infty. \tag{132}$$

Since the quantity in parentheses is always less than 1 in this case of $\beta > \omega$, we have $b_n \to 0$ as $n \to +\infty$.

For $\beta < \omega$, we use [7, theorem 8.22.7] and find now that

$$b_n \propto \exp(\sqrt{2n}\ A) \left(\frac{\sqrt{\omega^2-\beta^2}}{\beta}\right)^n\ n^{-1/4} \quad \text{as } n \to +\infty, \tag{133}$$

where A is the absolute value of the argument of He_n in (130). This quantity (133) tends to zero with n, regardless of A, when $\beta > \omega/\sqrt{2}$.

Combining with the result above, we can conclude that

$$b_n \to 0 \text{ as } n \to +\infty \qquad \text{for } \frac{\omega}{\sqrt{2}} < \beta < +\infty. \tag{134}$$

Furthermore, b_n behaves as an n-th power, which is faster than $n^{-1/4}$, thereby guaranteeing convergence of the probability density function and cumulative distribution function series, according to the discussion in (50A) et seq. On the other hand, $\{b_n\}$ diverges when $0 < \beta < \omega/\sqrt{2}$, as may be seen from (133).

The error integral in (21) is, for Hermite weighting (40) and probability density function (127),

$$\int du \, \frac{p^2(u)}{w(u)} = \frac{\beta^2}{\omega\sqrt{2\beta^2-\omega^2}} \exp\left(\frac{(\gamma-\alpha)^2}{2\beta^2-\omega^2}\right) \quad \text{if} \quad \frac{\omega}{\sqrt{2}} < \beta \ , \tag{135}$$

by use of [8, 3.323 2]; this integral is divergent if $\beta < \omega/\sqrt{2}$. Thus, for this particular example, the error integral and expansion coefficient sequence $\{b_n\}$ converge or diverge together, depending on the condition $\beta \gtrless \omega/\sqrt{2}$. The choice of α is irrelevant in this case.

A numerical example of sequence $\{b_n\}$ for

$$\gamma = 1.1, \ \omega = 2.3 \qquad \alpha = 1.14, \ \beta = 2.34 \tag{136}$$

is plotted in figure 1 on a logarithmic ordinate. Values of b_n less than 1E-7 in absolute value are all plotted at the ±1E-7 line. The critical ratio $\sqrt{\beta^2-\omega^2}/\beta$ in (130) is .184 for this example, leading to rapid decay of expansion coefficients $\{b_n\}$. The three sets of expansion coefficients in figure 1 are labelled according to the shorthand notation

RC: Recursively via Cumulants,

DM: Directly via Moments,

RM: Recursively via Moments. (137)

It is seen that the expansion coefficients determined recursively via cumulants, namely, the RC plot, decay rapidly and never encounter round-off error, whereas the DM and RM procedures both are subject to large round-off error for n > 70, as indicated by the large increasing oscillations. This example can be rather misleading, however, since all the cumulants (128A) of Gaussian probability density function (127A) are zero, except for $\chi_1 = \gamma$, $\chi_2 = \omega^2$; this leads to a very special form of the RC procedure unique to the Gaussian case.

In figure 2, the cumulative distribution function and exceedance distribution function, 1-P(u), as determined by Hermite expansion (48) using N = 50 terms, are plotted. The exact result, (127B), overlapped these curves over the full range plotted. The three procedures, RC, DM, and RM, all yielded identical distributions in figure 2, as inspection of figure 1 confirms, since the three sets of expansion coefficients are virtually the same for n < 50. Even though the three sets of expansion coefficients differ significantly for n > 60, the corresponding approximate probability density

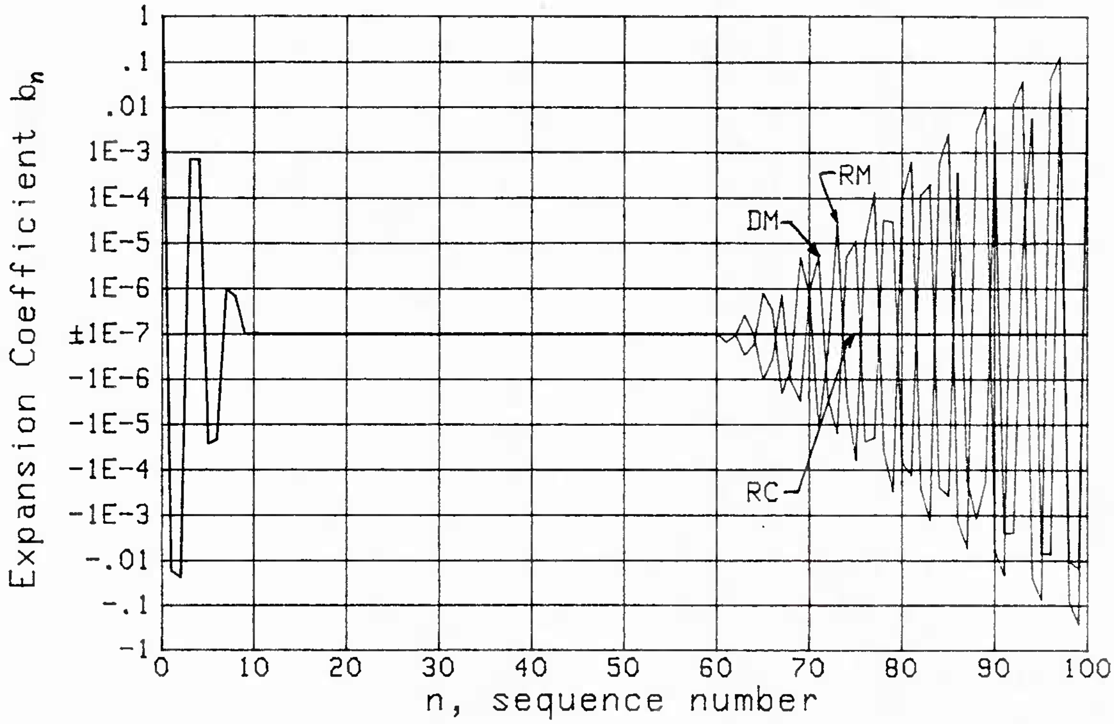

Figure 1. Hermite Coefficients for Example A

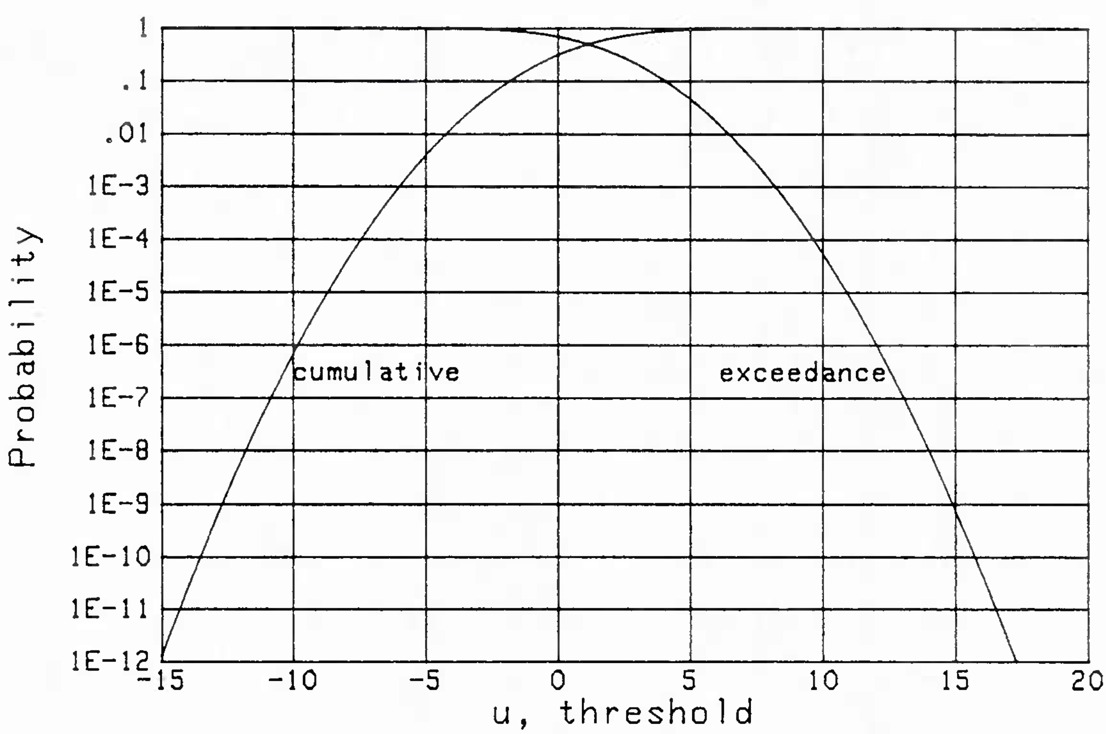

Figure 2. Distributions for Example A

functions and cumulative distribution functions for N = 70, say, would not be very different, because the relative differences in p and P are very small, somewhere in the 1E-5 range; see figure 1 for n = 70, and recall that $b_0 = 1$ for this example.

EXAMPLE B

The probability density function of interest here is the one previously considered in (34) et seq.:

$$p(u) = \frac{2\, u^{\gamma} \exp(-u^2/\omega^2)}{\omega^{\gamma+1}\, \Gamma\left(\frac{\gamma+1}{2}\right)} \quad \text{for } u > 0 \qquad (\gamma > -1,\ \omega > 0) \ . \tag{138}$$

This class of probability density functions includes, for $\gamma = 0, 1, 2$, respectively, the one-sided Gaussian, Rayleigh, and Maxwell as special cases. The characteristic function and cumulants are not easily determined directly for this function. However, the moments, as given already in (35), are readily evaluated via the simple recursion

$$\mu_n = \mu_{n-2}\, \frac{\omega^2}{2}\, (\gamma-1+n) \quad \text{for } n \geq 2, \quad \mu_0 = 1, \quad \mu_1 = \omega\, \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} \ . \tag{139}$$

An example of the expansion coefficients for

$$\gamma = 3,\ \omega = 1 \qquad \alpha = 0,\ \beta = .7 \tag{140}$$

is depicted in figure 3. The values of b_n for n = 0, 1, 2, 3 are 1, 1.90, 2.18, 1.63, respectively, and lie above the top of the plotted region. The

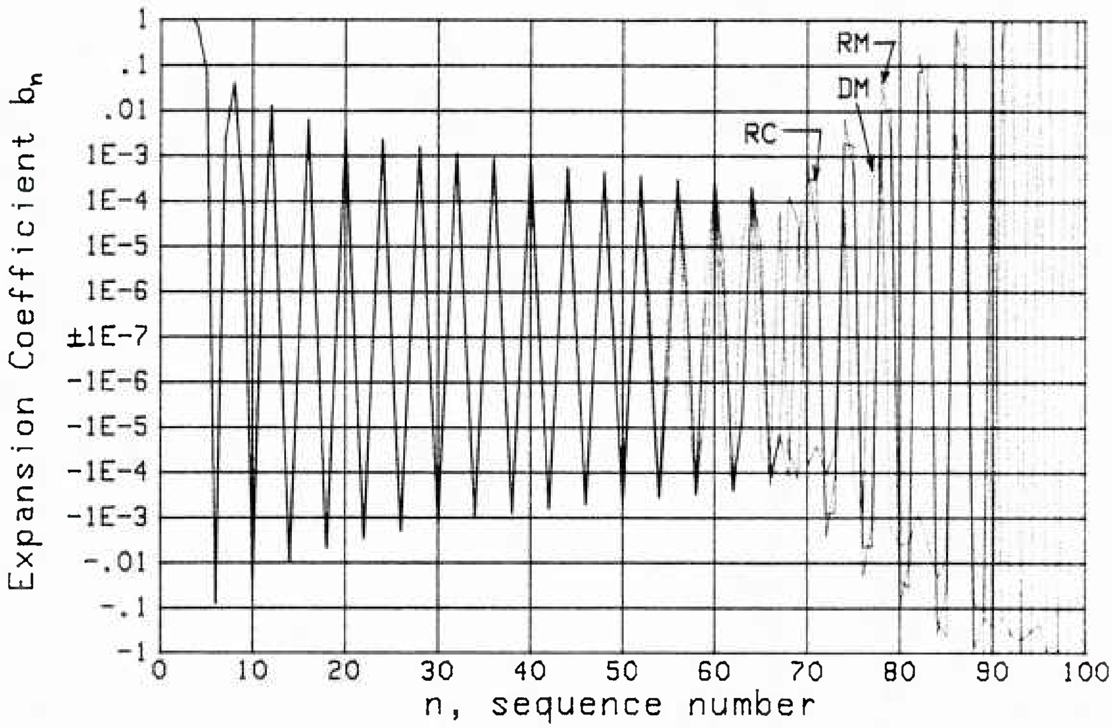

Figure 3. Hermite Coefficients for Example B

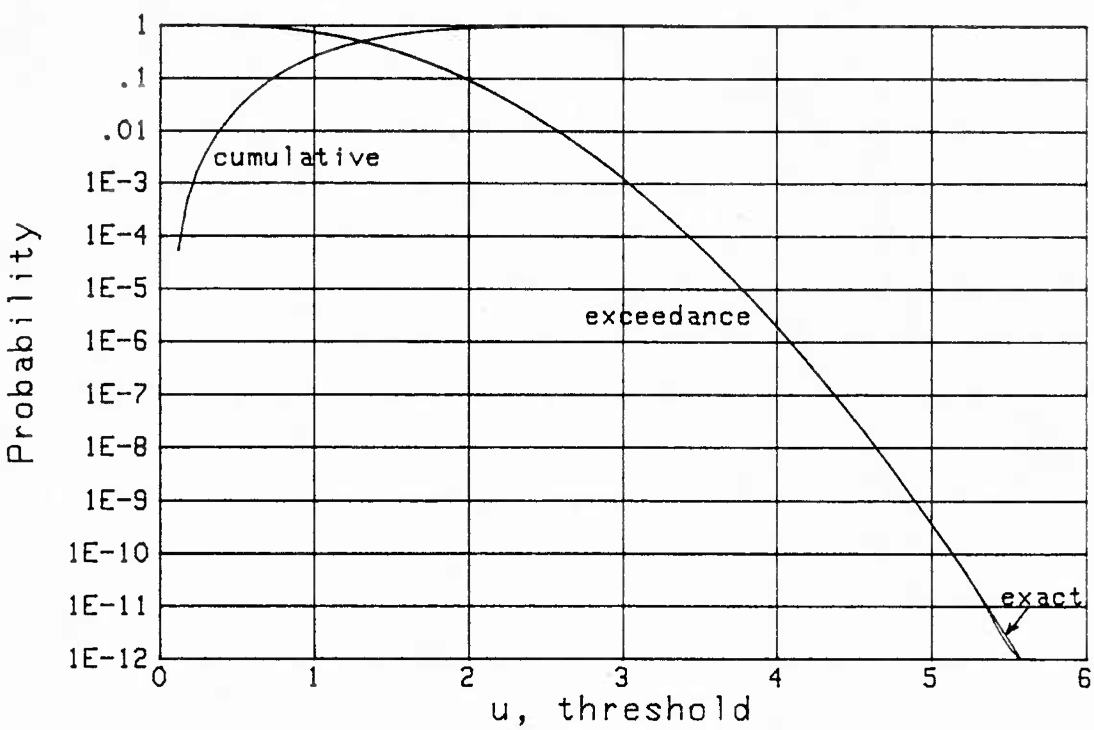

Figure 4. Distributions for Example B

coefficients obtained directly via moments, DM, decay to approximately 1E-4 near n = 70 and then encounter round-off error. The expansion coefficients corresponding to RC and RM are more noisy. The procedure used for RC was to determine the moments via (139), transform directly to cumulants according to (A-7), and then use (63).

A plot of the distributions using N = 65 terms is given in figure 4; the results are the same for all three sets of expansion coefficients, as may be seen by reference to figure 3. Furthermore, the exact cumulative distribution function, $P(u) = 1 - (1+u^2)\exp(-u^2)$ for $u > 0$, overlays these results except for the bow in the exceedance distribution function below 1E-11 near $u = 5.5$. Values of the cumulative distribution function for $u < 0$ as determined by series (48) are not zero, although they should be for this example; the generalized Laguerre series would fit this example better, since it is nonzero only for positive arguments.

EXAMPLE C

Consider the class of Bessel-function probability density functions

$$p(u) = \Lambda\, u^{\gamma} \exp(-u^2/\omega^2)\; I_{\zeta}(\theta u) \quad \text{for } u > 0 \;, \tag{141}$$

which includes the Rice and generalized Q_M distributions, for example. The n-th moment is [8, 6.631 1]

$$\mu_n = \Lambda \frac{\theta^{\zeta}\, \Gamma\!\left(\frac{n}{2}+h\right)\; \omega^{n+2h}}{2^{\zeta+1}\, \Gamma(\zeta+1)}\; {}_1F_1\!\left(\frac{n}{2}+h; \zeta+1; \frac{1}{4}\,\omega^2\theta^2\right) \quad \text{for } n \geq 0 \;, \tag{142}$$

with $h = (\gamma+\delta+1)/2$; in order for μ_0 to be finite, we must have $h > 0$. The $_1F_1$ function in (142) can be evaluated via recursion; this leads to a recursion for the moments (see appendix E).

We consider here only the special case of the Rice probability density function, namely,

$$\Lambda = \frac{2}{\omega^2} \exp\left(-\frac{1}{4}\,\omega^2\theta^2\right) , \quad \gamma = 1, \quad \delta = 0 , \tag{143}$$

for which

$$p(u) = \frac{2u}{\omega^2} \exp\left(-\frac{u^2}{\omega^2} - \frac{\omega^2\theta^2}{4}\right) I_0(\theta u) \quad \text{for } u > 0 . \tag{144}$$

The moments in (142) then reduce to

$$\mu_n = \Gamma\left(\frac{n}{2}+1\right) \omega^n \; {}_1F_1\left(-\frac{n}{2};1;-\frac{1}{4}\,\omega^2\theta^2\right) , \tag{145}$$

and can be easily determined by the recurrence presented in (E-5). The cumulative distribution function corresponding to (144) is the Q function [1]

$$P(u) = 1 - Q\left(\frac{\omega\theta}{\sqrt{2}} , \frac{\sqrt{2}\,u}{\omega}\right) \quad \text{for } u > 0 ; \tag{146}$$

the characteristic function is given in [9, appendix A] as an infinite series, meaning that the cumulants cannot be determined directly, except via the moments.

The particular example we consider here for the Hermite expansion is a sum of 8 independent random variables, each with Rice probability density function (144). For direct comparison with the exact results in [9], we also consider the normalized form of (144), namely $\omega^2 = 2$. Furthermore, we limit numerical consideration in this particular example to evaluation of the cumulative and exceedance distribution functions for $\theta = 0$, which corresponds physically to the false alarm probability for the sum of eight normalized envelopes of narrowband Gaussian noise (i.e., a Rayleigh probability density function for the individual random variables).

For $\alpha = 4$, $\beta = 2.15$, the expansion coefficients $\{b_n\}$ are displayed in figure 5 for the RC, DM, and RM approaches. All the $\{b_n\}$ for $1 \leq n \leq 20$ are bigger than 1; the biggest is $b_6 = 12.25$. The $\{b_n\}$, for both moment approaches, have not been plotted for $n > 60$ because they continue to oscillate well beyond the ± 1 limits, while the RC coefficients decay exponentially with n. Despite the fact that the moments were the initially determined quantities for this example, the RC method far outperforms the DM and RM methods, as seen in figure 5. The reason for this is as follows: for the RC method, the procedure was to obtain moments via (145), cumulants via (A-7), cumulants of the sum of 8 independent random variables by simple scaling by a factor of 8, and then expansion coefficients via (63). For the DM and RM methods, the moments of the sum of 8 random variables were determined via [6, (14)] which progressively determined the moments of a sum of 2 random variables, then 3, 4,..., 8 in order, and then employed (70). This iterated procedure for moments requires more number-crunching and leads to considerably larger round-off error than the simple scaling required for the RC procedure. Thus it appears that when the random variable of interest is obtained as a sum of several independent random variables, the RC approach will be the prime candidate for expansion coefficient evaluation; this applies also if the individual random variables have different statistics, but remain independent.

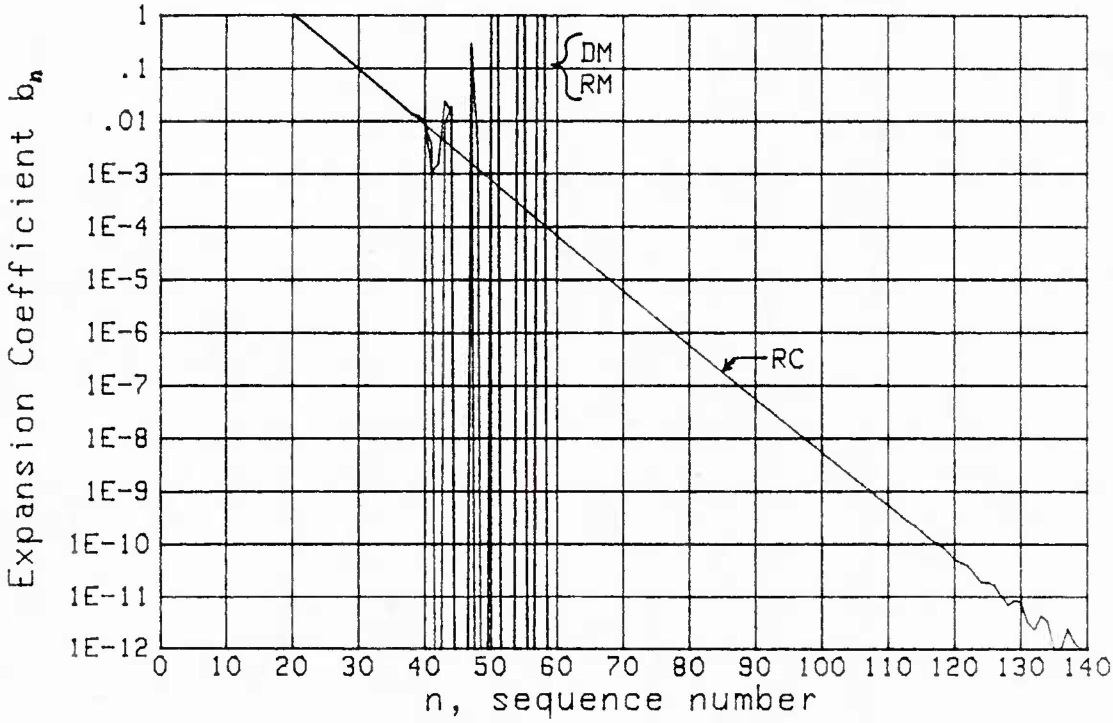

Figure 5. Hermite Coefficients for Example C

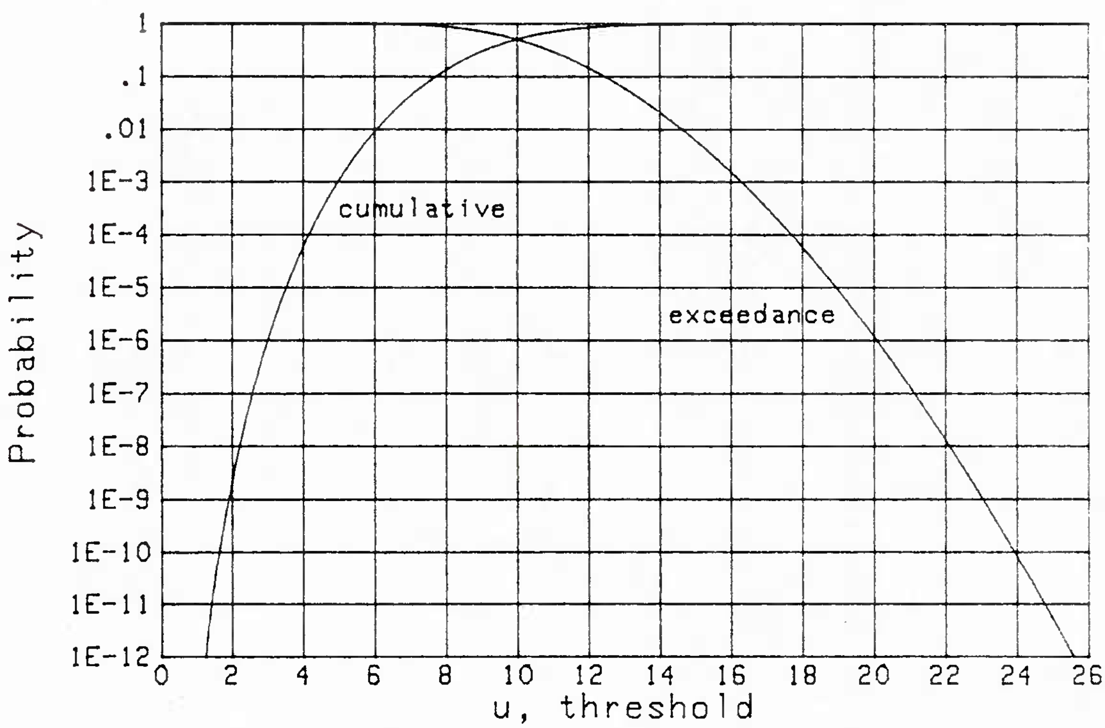

Figure 6. Distributions for Example C

The cumulative and exceedance distribution functions for this sum of 8 normalized Rayleigh variates are plotted in figure 6, for the N = 140 expansion coefficients of the RC procedure in figure 5. In order to make a precise determination of the accuracy of this Hermite series approach, the false alarm probabilities were computed at the eight thresholds listed under M = 8 in [9, table 1]. To the precision given in that table, the computed probabilities were exactly the specified values 1E-m for m = 1(1)8. Thus, as anticipated by figure 5, very accurate evaluation of false alarm probabilities are possible by this series approach.

A short search of values of the best weighting parameters α and β, to use with the DM approach, led to $\alpha = 5.84$, $\beta = 2.28$ and expansion coefficients b_n near 1E-4 at n = 28, before round-off error became dominant. This is better than the result of DM in figure 5 for $\alpha = 4$, $\beta = 2.15$. Evaluation of the false alarm probabilities at the thresholds in [9, table 1] gave 7 decimal accuracy at .1, and 4 decimal accuracy at 1E-8. This is adequate for most purposes, but is not as good as the RC approach.

EXAMPLE D

In [4, appendix C], the characteristic function for shot noise with random amplitude and duration modulation, and arbitrary individual pulse shape, is derived. (This result is then specialized to elliptical pulses and Rayleigh amplitude modulation [4, (C-36)-(C-42)].) Also, the cumulants are extracted, with general result [4, (24)], where ν is the average number of pulses/second, $\overline{\ell}$ is the average length of the duration modulation, $\mu_a(n)$ is the n-th moment of the amplitude modulation, and F(x) is the individual pulse shape of the shot noise. Thus shot noise is a case where the cumulants are directly capable of evaluation, whereas the moments must be found indirectly.

For the special case of elliptical pulses and Rayleigh amplitude modulation, there follows for the cumulants [4, (29)]:

$$\chi_n = \nu\bar{\ell}\,\sigma_a^n\,2^{\frac{3}{2}n+1}\,\Gamma^3\!\left(\tfrac{n}{2}+1\right)\Big/\Gamma(n+2) \quad \text{for } n \geq 1, \quad \chi_0 = 0 . \tag{147}$$

These quantities are easily evaluated via recurrence

$$\chi_n = \chi_{n-2}\,\sigma_a^2\,n^2/(n+1) \quad \text{for } n \geq 3,\ \chi_1 = \left(\frac{\pi}{2}\right)^{3/2}\nu\bar{\ell}\,\sigma_a,\ \chi_2 = \frac{8}{3}\,\nu\bar{\ell}\,\sigma_a^2 . \tag{148}$$

This procedure was used in [4, appendix D] to obtain the probability density function and cumulative distribution function results given there.

There is a nuance that arises in shot noise for pulse shapes of finite duration; see [4, pp. 40-42]. Namely, there is an impulse in the probability density function, at u = 0, of area

$$P_0 = \exp[-\nu\bar{\ell}\,(x_2 - x_1)] , \tag{149}$$

where (x_1,x_2) is the non-zero extent of an unmodulated individual pulse. Since an impulse is very difficult to approximate by a finite series of continuous functions, the effect of this quantity should be subtracted from the statistics (moments or cumulants), and the continuous portion of the probability density function should be approximated. Similarly, the corresponding step in the cumulative distribution function at the origin should be eliminated from the approximation procedure.

This feature is easily incorporated if P_0 is subtracted from the zero-th order moment [4, p. 42]. The only undesireable side-effect of this manipulation is that the initially computed cumulants must be transformed to moments, then μ_0 corrected, and then all the new cumulants evaluated. This double transformation is necessary because the correction (subtraction) procedure can only be accomplished in the moment domain. Of course, when the DM or RM procedures are employed instead of RC, the last transformation to cumulants is unnecessary; this was, in fact, the procedure used in [4, p. 60].

When the individual pulse F(x) has infinite duration, as for an exponential or Gaussian waveform, then $x_2 - x_1$ is infinite and P_0 in (149) is zero. In that case, the considerations in the last two paragraphs can be disregarded, and the cumulants generated via (148) used as is. It is then very likely that even better accuracy in the expansion coefficients will be achieved than for this current example.

For overlap factor [4, p. 43]

$$\overline{K_1} = \nu\bar{\ell}(x_2 - x_1) = 6.2, \quad P_0 = \exp(-6.2) = .00203, \quad \sigma_a = 1 , \qquad (150)$$

and for weighting parameters $\alpha = 6.1$, $\beta = 4.3$, the expansion coefficients $\{b_n\}$ are displayed in figure 7 for the three recursive procedures. The RM results are considerably poorer than the RC and DM coefficients, which are

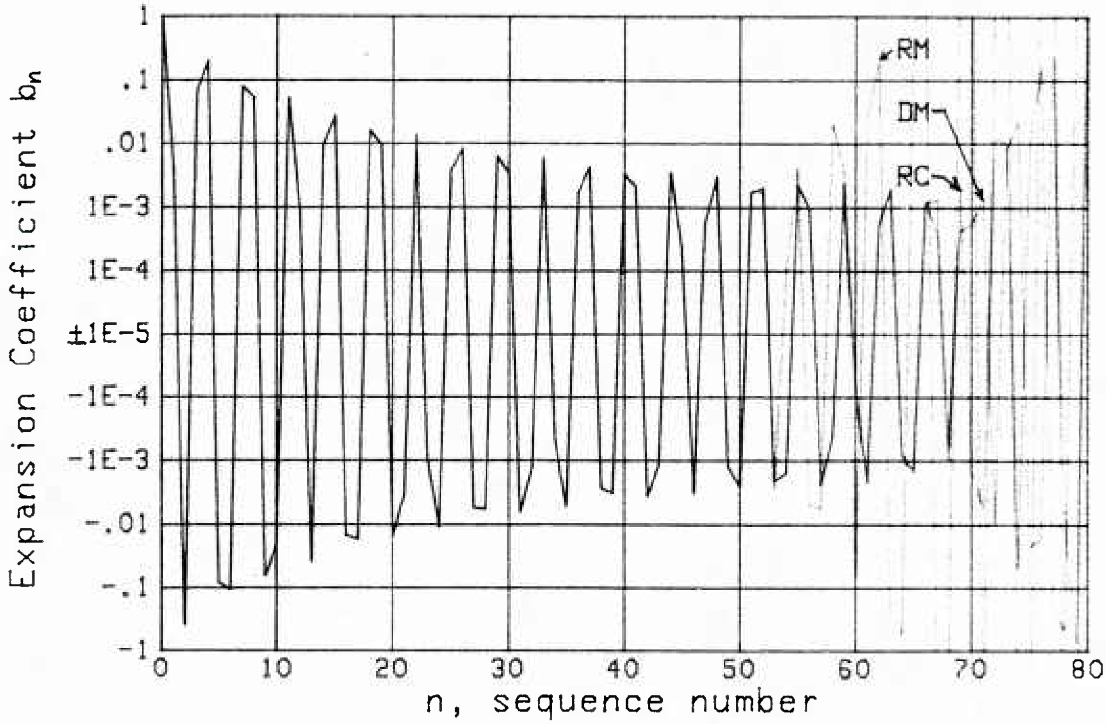

Figure 7. Hermite Coefficients for Example D

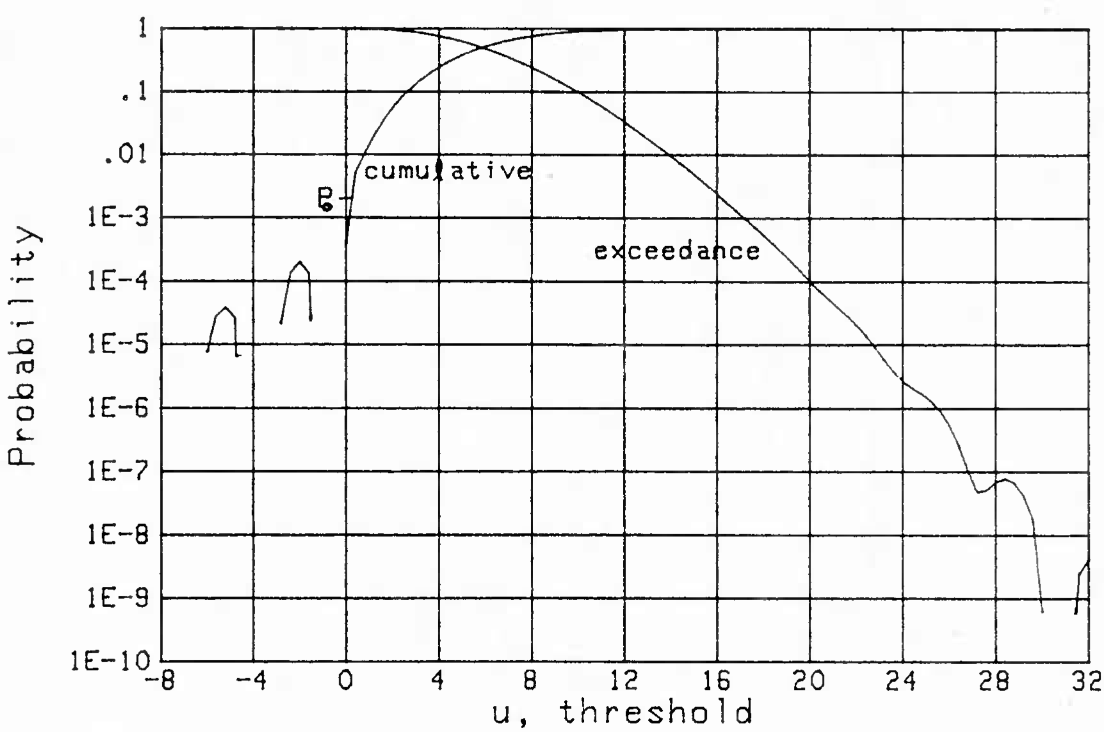

Figure 8. Distributions for Example D

comparable for $n < 65$. However, even here, the coefficients have only decayed to the 1E-3 level, which may not be sufficiently small for accurate results.

The distributions using $N = 65$ terms and the RC expansion coefficients are given in figure 8. Although the actual cumulative distribution function is zero for $u < 0$, the approximation oscillated around zero, reaching a positive peak of value .22E-3 at $u = -2$. Similarly, significant wiggles develop in the exceedance distribution function below the 1E-4 level. The reason for the inadequacy of these Hermite expansions near $u = 0$ is the abrupt zero behavior of the true probability density function for negative arguments, a feature inherently difficult to approximate by means of smooth continuous functions. The error of the approximations in figure 8 is estimated in a later section and superposed on the plot, for ease of ascertaining the reliability of the curves. The corresponding approximations for the generalized Laguerre series are better for this type of probability density function, as will be demonstrated in the next section.

The approximate probability density function for this example, again with $N = 65$ terms, is given in figure 9 on a linear ordinate. It reaches a negative peak of -8E-4, and crosses the $u = 0$ axis with value .004; both of these values should be zero, and will be for the generalized Laguerre series. To see how the approximate probability density function behaves for larger arguments, the logarithmic plot in figure 10 is used. Wiggles develop near the 1E-4 level and become large enough that negative values of the density are yielded near $u = 28$ and 31. It will be worthwhile to compare this Hermite series with the generalized Laguerre series to be presented in the next section. The estimated error associated with figure 10 is developed in a later section.

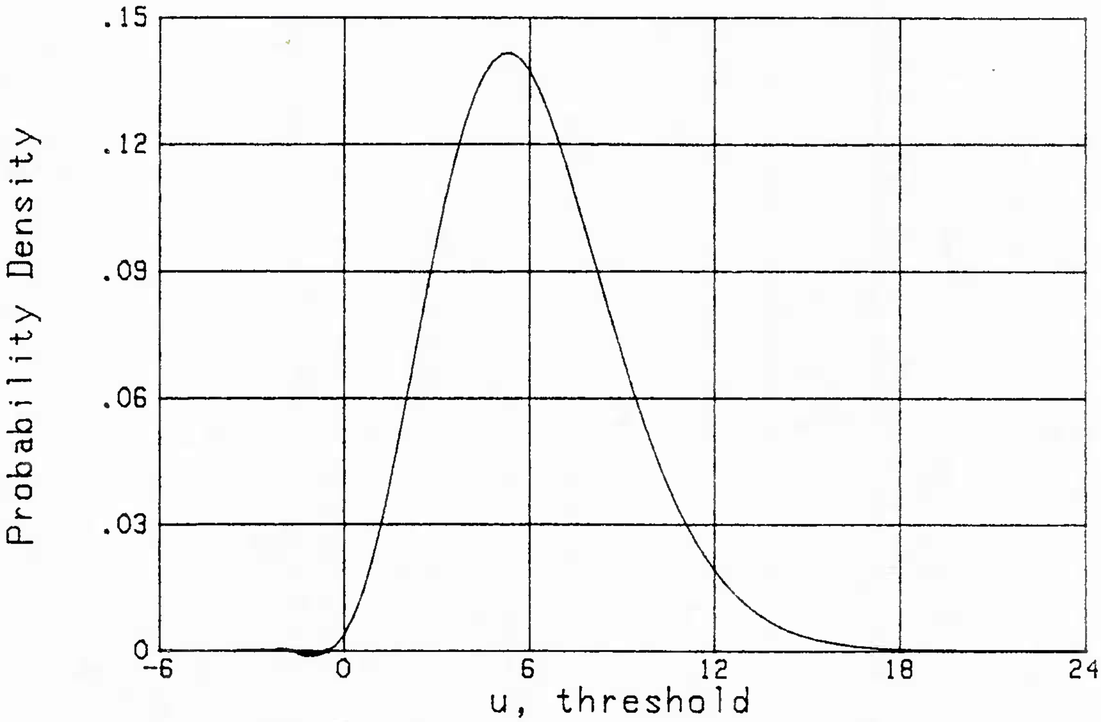

Figure 9. Linear Density for Example D

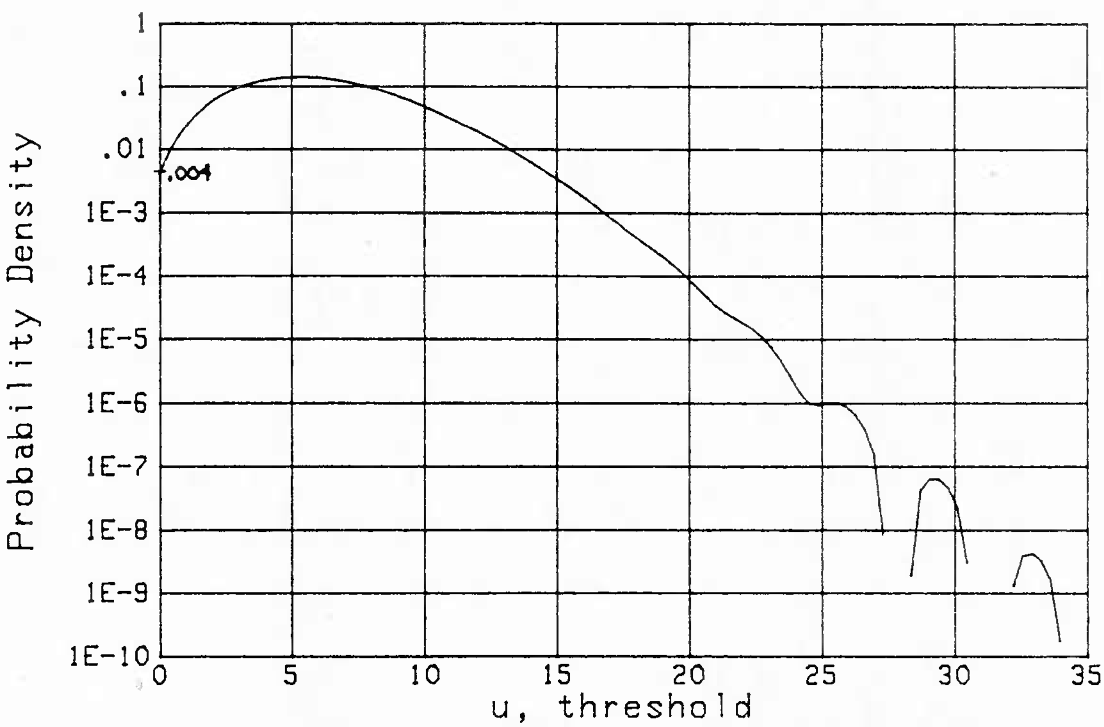

Figure 10. Log Density for Example D

EXAMPLES OF GENERALIZED LAGUERRE EXPANSION

EXAMPLE E

As with the earlier Hermite expansions, the first generalized Laguerre example here is one that can be evaluated analytically, for purposes of checking numerical procedures and results. Namely consider the Chi-square probability density function of $2(\gamma+1)$ degrees of freedom (which need not be integer):

$$p(u) = \frac{u^{\gamma} \exp(-u/\omega)}{\omega^{\gamma+1}\,\Gamma(\gamma+1)} \qquad (\gamma > -1,\ \omega > 0)\ . \tag{151}$$

All probability density functions and approximations are limited to $u > 0$ in this section, since they are zero for $u < 0$; this restriction will be presumed in the remainder of the presentation.

The exceedance distribution function is related to the incomplete Gamma function [5, 6.5.3]:

$$1 - P(u) = \int_u^\infty dt\ p(t) = \Gamma(\gamma+1,\ u/\omega) / \Gamma(\gamma+1)\ . \tag{152}$$

The characteristic function follows from (151) as

$$f(i\xi) = (1 - i\xi\omega)^{-\gamma-1}\ , \tag{153}$$

with cumulants

$$\chi_k = (k-1)!\ (\gamma+1)\ \omega^k \quad \text{for } k \geq 1,\ \ \chi_0 = 0\ , \tag{154}$$

and moments

$$\mu_k = (\gamma+1)_k\ \omega^k \quad \text{for } k \geq 0\ . \tag{155}$$

Thus either set of statistics can be used as a starting position. The error integral in (21) is finite if

$$-1 < \alpha < 2\gamma + 1 \quad \text{and} \quad \beta > \omega/2 \,. \tag{156}$$

We will find the expansion coefficients by means of the characteristic function expansion (100), developed earlier for the generalized Laguerre series. Specifically, we utilize the power series expansion

$$(1-w)^{-\alpha-1} f\left(\frac{-w/\beta}{1-w}\right) = (1-w)^{\gamma-\alpha} \left(1-w\frac{\beta-\omega}{\beta}\right)^{-\gamma-1} =$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha-\gamma)_m}{m!} w^m \sum_{k=0}^{\infty} \frac{(\gamma+1)_k}{k!} \left(\frac{\beta-\omega}{\beta}\right)^k w^k \,, \tag{157}$$

where we used (153) and [5, 15.1.8] twice. The coefficient of a general term w^n is then immediately given by the closed form

$$c_n = \sum_{m=0}^{n} \frac{(\alpha-\gamma)_m}{m!} \frac{(\gamma+1)_{n-m}}{(n-m)!} \left(\frac{\beta-\omega}{\beta}\right)^{n-m} \quad \text{for } n \geq 0 \,. \tag{158}$$

Alternative expressions for the expansion coefficients are

$$c_n = \frac{(\gamma+1)_n}{n!} \left(\frac{\beta-\omega}{\beta}\right)^n F\left(\alpha-\gamma, -n; -n-\gamma; \frac{\beta}{\beta-\omega}\right) =$$

$$= \frac{(\gamma+1)_n}{n!} \left(\frac{-\omega}{\beta}\right)^n F\left(-n, -n-\alpha; -n-\gamma; \frac{\beta}{\omega}\right) =$$

$$= \frac{(\alpha+1)_n}{n!} F\left(-n, \gamma+1; \alpha+1; \frac{\omega}{\beta}\right) \quad \text{for } n \geq 0 \,, \tag{159}$$

obtained by means of [5, 15.1.1, 15.3.5, 15.3.7] respectively. In fact, the last result can be obtained directly by using [8, 7.414 7] on (90) and (151):

$$c_n = \int_0^\infty du \, \frac{u^\gamma \exp(-u/\omega)}{\omega^{\gamma+1}\,\Gamma(\gamma+1)} \, L_n^{(\alpha)}\left(\frac{u}{\beta}\right) . \tag{160}$$

However, the latter two results in (159) are not numerically stable, whereas (158) and the first line of (159) are stable for large n, without encountering round-off error.

Some special cases of (158) are as follows:

$$\text{if } \alpha = \gamma, \text{ then } c_n = \frac{(\gamma+1)_n}{n!}\left(\frac{\beta-\omega}{\beta}\right)^n ;$$

$$\text{if } \beta = \omega, \text{ then } c_n = \frac{(\alpha-\gamma)_n}{n!} ;$$

$$\text{if } \alpha = \gamma \text{ and } \beta = \omega, \text{ then } c_n = \delta_{n0} . \tag{161}$$

The last case is to be expected, since the weighting exactly matches the probability density function (151) then.

A numerical example of sequence $\{b_n\}$ for

$$\gamma = 1.1, \quad \omega = 2.3 \qquad \alpha = 1.105, \quad \beta = 2.1 \tag{162}$$

is shown in figure 11, using the three recursive procedures developed earlier for the generalized Laguerre series in (112), (119), (126). In addition, exact result (158) is plotted for comparison. The expansion coefficients have a rapidly decaying transient for $n < 10$, and then a decay approximately proportional to $n^{-3/2}$ for large n. The abrupt change of character at $n = 5$ does <u>not</u> signify the onset of round-off error; rather, the latter is indicated

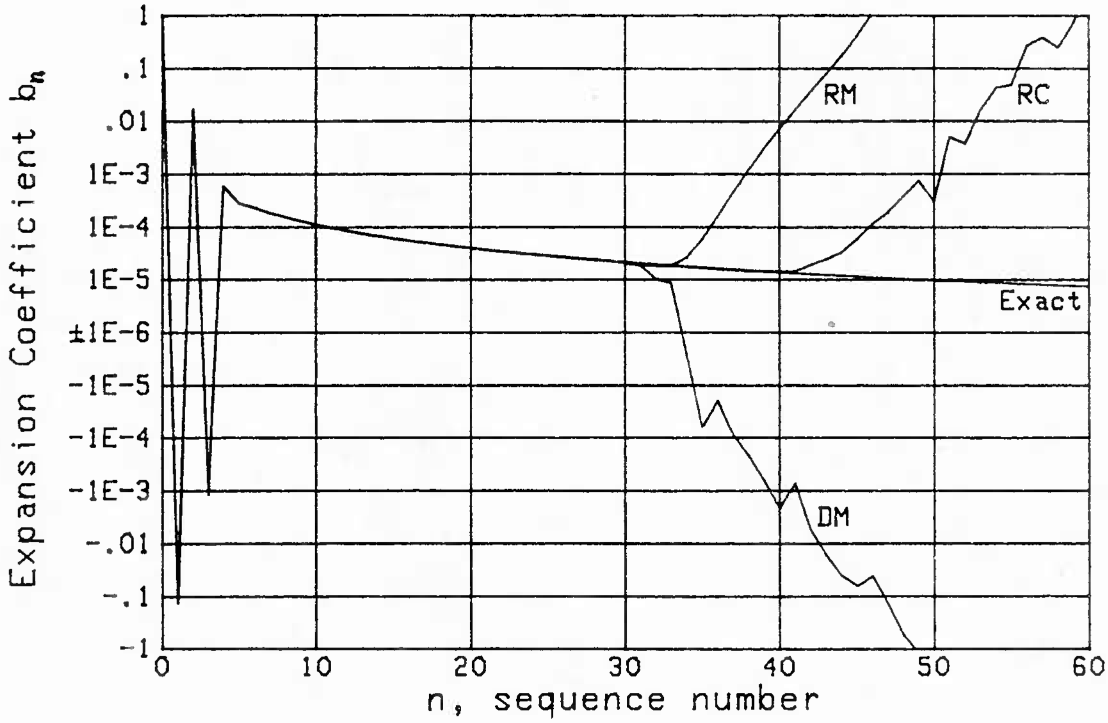

Figure 11. Generalized Laguerre; Example E

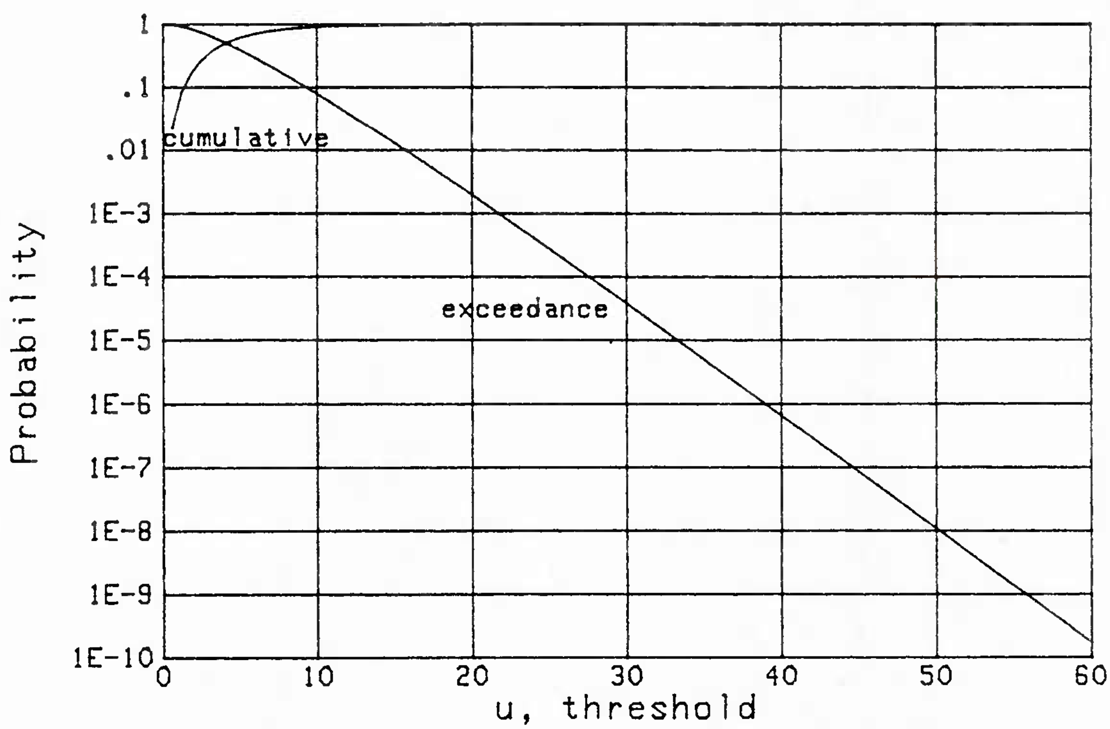

Figure 12. Distributions for Example E

by an erratic behavior, typically increasing exponentially with n (linear growth on a logarithmic ordinate).

A different plotting strategy will be adopted henceforth for the expansion coefficients $\{b_n\}$, in order not to clutter the diagrams with large oscillations as in figures 1, 3, 5, 7. Specifically, when the expansion coefficient b_n first exceeds the ± 1 limits, the remainder of sequence $\{b_n\}$ will not be plotted, since this is a region of large round-off error. Thus, although the RM curve in figure 11 returns to the ± 1 limits briefly at $n = 52,53$, these values are not displayed.

Round-off error for the RC procedure does not become as significant as for the two moment approaches until n has increased by almost 10, for this example in figure 11. In fact, the expansion coefficients for the RC procedure overlap the exact values until $n = 40$. The corresponding approximate distributions, using $N = 40$ terms in expansion (95) as determined by RC, are plotted in figure 12. The exact result (152) overlays these results over the entire range plotted.

EXAMPLE F

The following probability density function corresponds to a noncentral Chi-square variate of 2ν degrees of freedom:

$$p(u) = \frac{1}{2} \exp\left(-\frac{d^2+u}{2}\right)\left(\frac{\sqrt{u}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{u}) \qquad (\nu > 0) \; ; \qquad (163)$$

d is the noncentrality parameter, and 2ν need not be integer. The characteristic function is [8, 6.631 4]

$$f(i\xi) = (1-i2\xi)^{-\nu} \exp\left(\frac{id^2\xi}{1-i2\xi}\right) , \tag{164}$$

and is the same as the one considered in [10, (50) et seq.]. The exceedance distribution function is the generalized Q-function:

$$1 - P(u) = \int_u^\infty dt \, \frac{1}{2} \exp\left(-\frac{d^2+t}{2}\right) \left(\frac{\sqrt{t}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{t}) =$$

$$= \int_{\sqrt{u}}^\infty dx \, x \exp\left(-\frac{d^2+x^2}{2}\right) \left(\frac{x}{d}\right)^{\nu-1} I_{\nu-1}(dx) = Q_\nu(d,\sqrt{u}) . \tag{165}$$

By expanding the ℓn of (164) in a power series in $i\xi$, the cumulants follow as

$$\chi_n = 2^n(n-1)! \left(\nu + \frac{1}{2} d^2 n\right) \quad \text{for } n \geq 1, \quad \chi_0 = 0 . \tag{166}$$

And the moments are obtained from (163) as

$$\mu_n = 2^n (\nu)_n \, {}_1F_1(-n;\nu;-d^2/2) =$$

$$= 2^n n! \, L_n^{(\nu-1)}(-d^2/2) \quad \text{for } n \geq 0 , \tag{167}$$

by use of [8, 6.631 1] and [5, 13.6.9]. Both (166) and (167) lend themselves to simple recurrences which involve only positive quantities; thus the starting statistics can be quickly and accurately evaluated.

The numerical example we consider here will be compared with the exact results in [10, figure 11], namely,

$$\nu = 2.7, \qquad d = 3 \qquad\qquad \alpha = 1.7, \qquad \beta = 5.5 \ . \tag{168}$$

Since the probability density function in (163) behaves as $u^{\nu-1}$ as $u \to 0+$, it is reasonable to choose weighting parameter α in (82) as ν-1, as indicated in (168). And since (163) behaves as $\exp(-u/2)$ as $u \to +\infty$, we must choose $\beta > 1$ in order that the error integral in (21) is finite. The particular values in (168) approximately minimize the sum of $\left\{b_n^2\right\}_0^N$ in (21).

The expansion coefficients $\left\{b_n\right\}$ as determined by the three available recursive procedures are displayed in figure 13. The RC coefficients decrease to values less than 1E-10 near n = 50, before round-off error becomes significant. The two moment approaches deteriorate near n = 30, which is markedly poorer than the cumulant approach. The distributions, as determined by N = 50 terms of the RC approach, are given in figure 14, and agree with the d = 3 curve of [10, figure 11]. When the approximate probability density function for N = 50 was compared with exact result (163), 10 decimals of agreement were obtained; this is due to the ability to get very small $\left\{b_n\right\}$ in figure 13 via the RC method.

EXAMPLE G

This example is the Rice probability density function given in (144), with moments (145) and cumulative distribution function (146). The starting statistics are the moments as determined by recurrence (E-5)-(E-6).

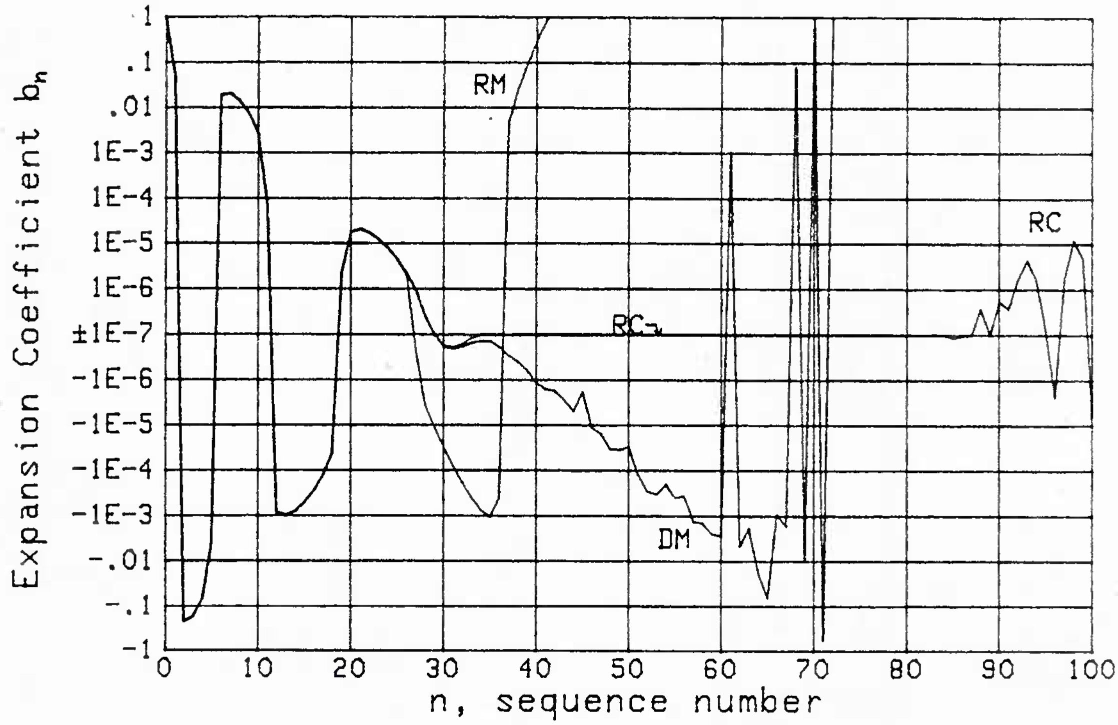

Figure 13. Generalized Laguerre; Example F

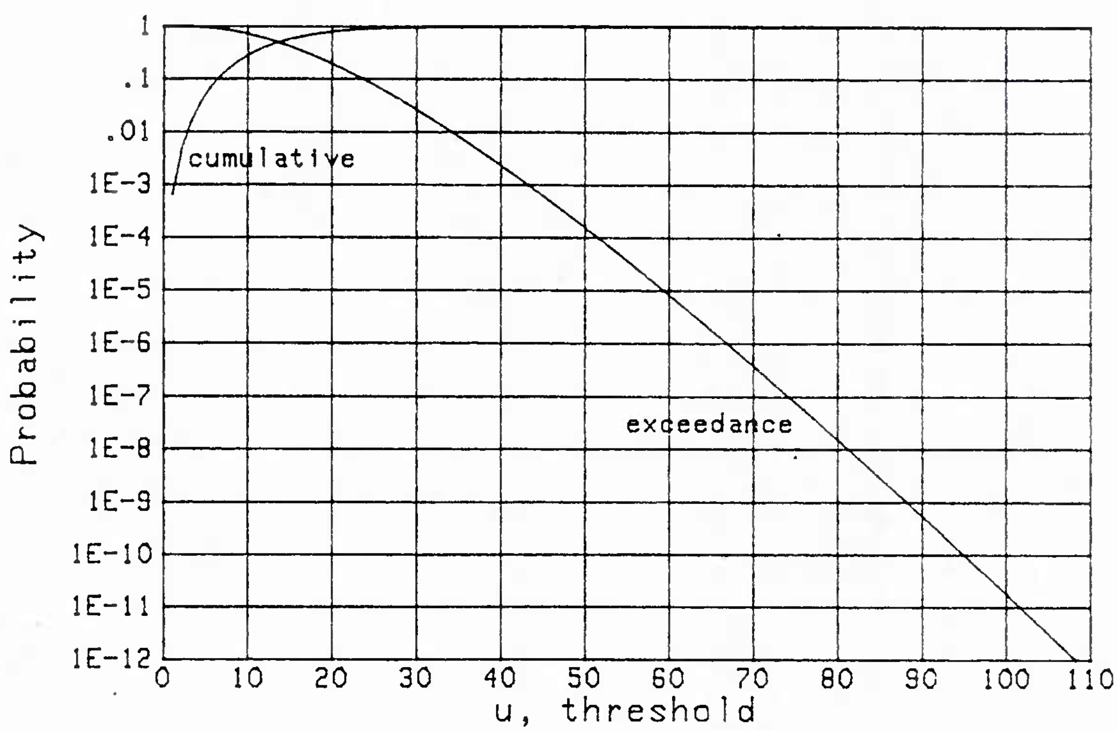

Figure 14. Distributions for Example F

The particular numerical case of interest is

$$\theta = 3, \qquad \omega^2 = 2 \qquad\qquad \alpha = 1, \qquad \beta = 1. \tag{169}$$

The values of α and β were found by the usual trial and error search procedure of observing plots of expansion coefficients $\{b_n\}$, looking for rapid decay and small round-off error; results for this example are displayed in figure 15. The RM procedure deteriorates rapidly at n = 30, whereas DM and RC are useable up to n = 55 and 65 approximately.

The cumulative and exceedance distribution functions for N = 65 terms of the RC procedure are plotted in figure 16, along with exact result (146). The approximate exceedance distribution function overlaps the exact one until slightly below the probability level 1E-4, which corresponds to the level of reliability of b_n in figure 15 at n = 65. Then the exceedance distribution function makes a positive (upward) turn below 1E-6, which is impossible for a physical density function which must remain positive; thus the approximation deteriorates rapidly for u > 7.

EXAMPLE H

This is a follow-on to the previous example, in that we consider a sum of 8 Rice variates, each with the statistics in (169). The expansion coefficients for

$$\theta = 3, \qquad \omega^2 = 2 \qquad\qquad \alpha = 26, \qquad \beta = 1 \tag{170}$$

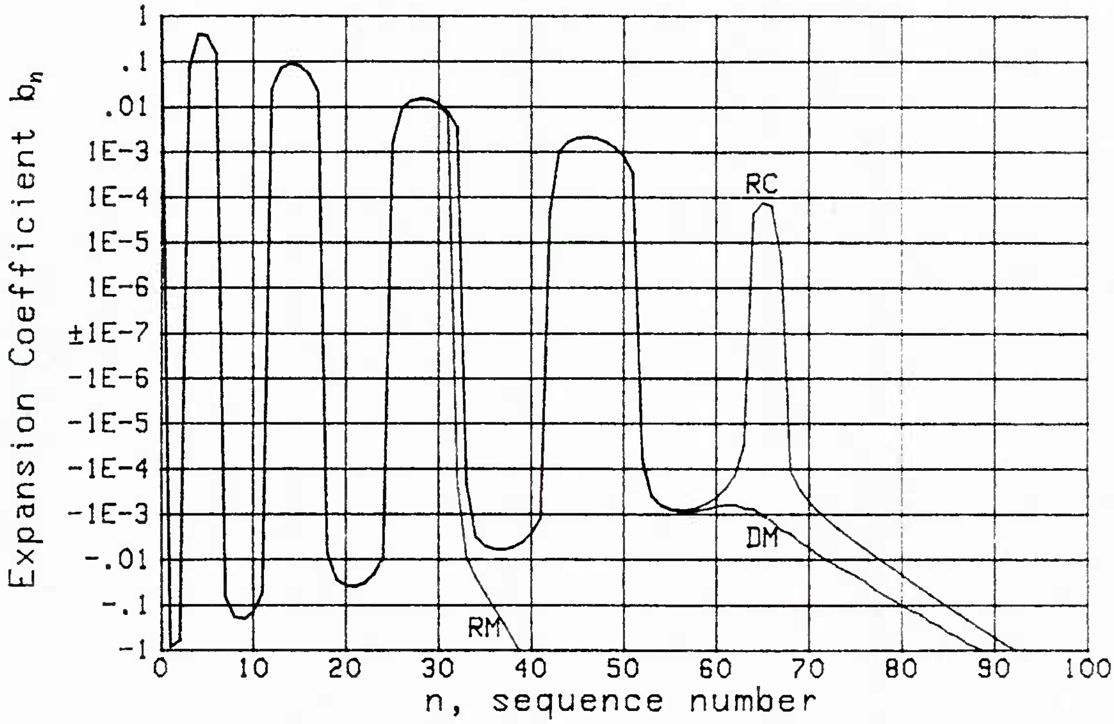

Figure 15. Generalized Laguerre; Example G

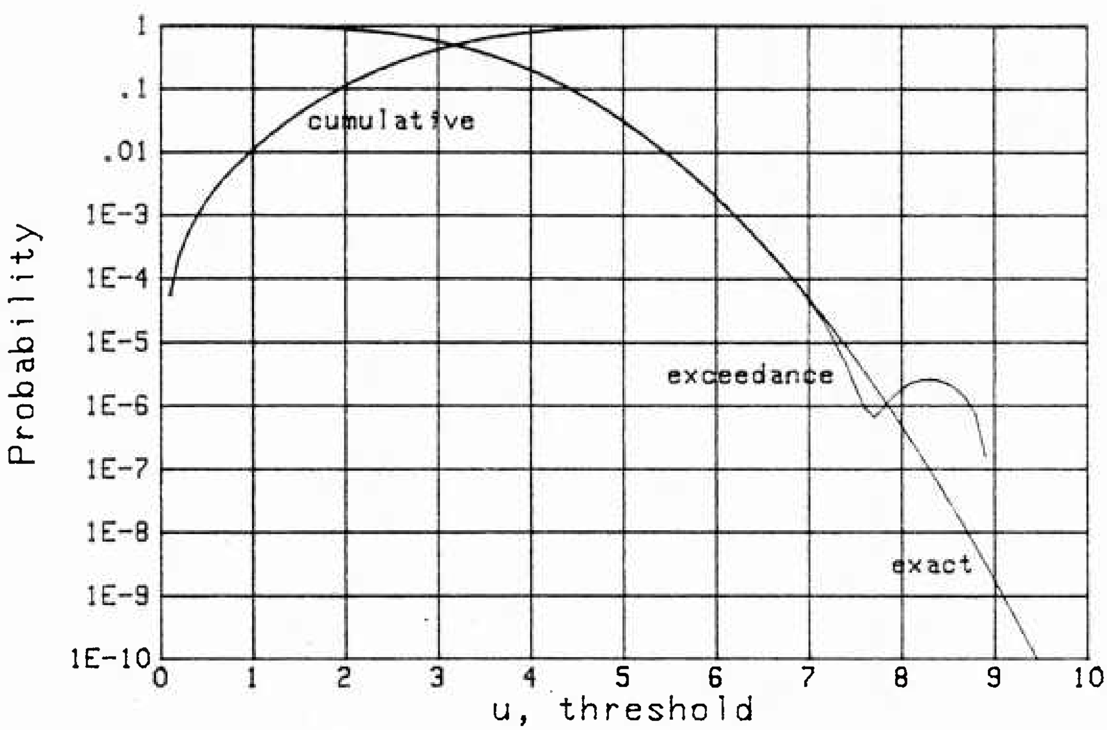

Figure 16. Distributions for Example G

are displayed in figure 17. Whereas both DM and RM are useless beyond $n = 25$, the expansion coefficients determined by RC decay down to the 1E-10 level at $n \cong 150$ before round-off error becomes significant. The corresponding distributions in figure 18, using $N = 143$ terms of the expansion via RC, reveal accurate results down to the 1E-12 level of probability, except for a slight flare in the exceedance distribution function below 1E-11.

We also checked the example of the sum of 8 normalized Rayleigh variates considered earlier via a Hermite series in example C. For $\alpha = 10$, $\beta = .9$, the expansion coefficients $\{b_n\}$ decayed to the 1E-11 level at $n = 100$ for the RC approach and agreed with the false alarm probabilities calculated exactly in [9, table 1] for $M = 8$. By contrast, the DM expansion coefficients were subject to significant round-off error by the time n reached 30, and were useless for small probability calculations.

EXAMPLE I

We return to the shot noise process previously considered via a Hermite series in example D. The equations and discussions there should be reviewed, since they are directly relevant to the generalized Laguerre expansion here. For the choice of parameters in (150), the selection of generalized Laguerre weighting parameters

$$\alpha = .74, \qquad \beta = 2.1 \tag{171}$$

leads to the expansion coefficients plotted in figure 19. The DM and RC results agree to $n = 32$, and then begin to diverge from each other. By way of

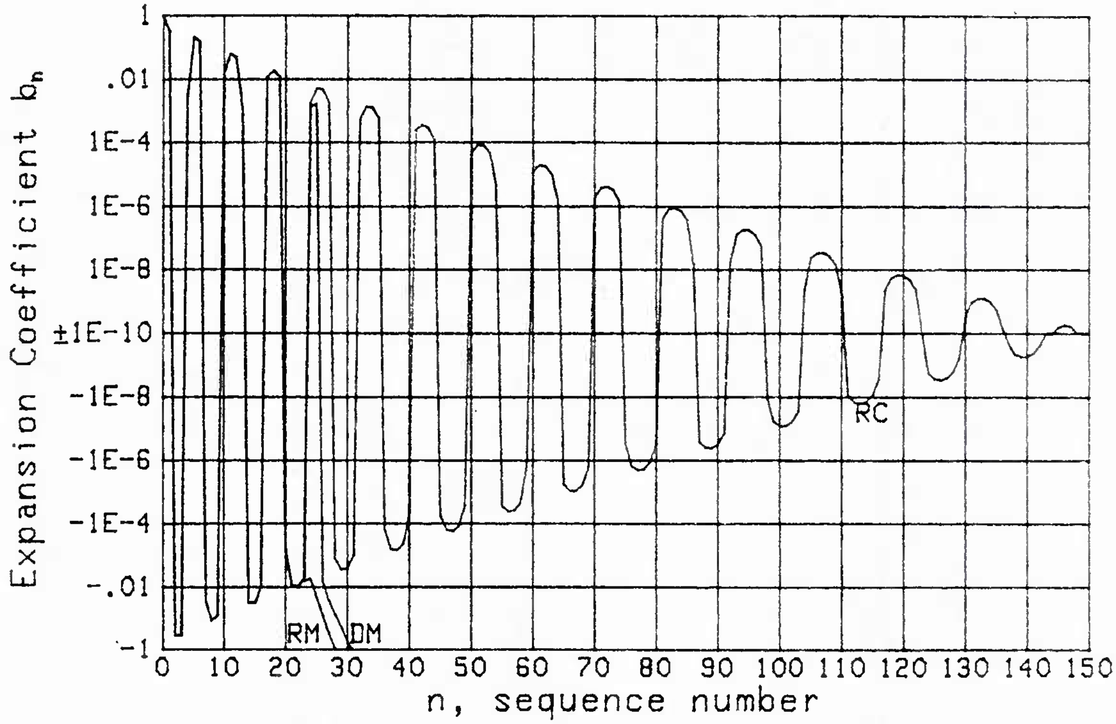

Figure 17. Generalized Laguerre; Example H

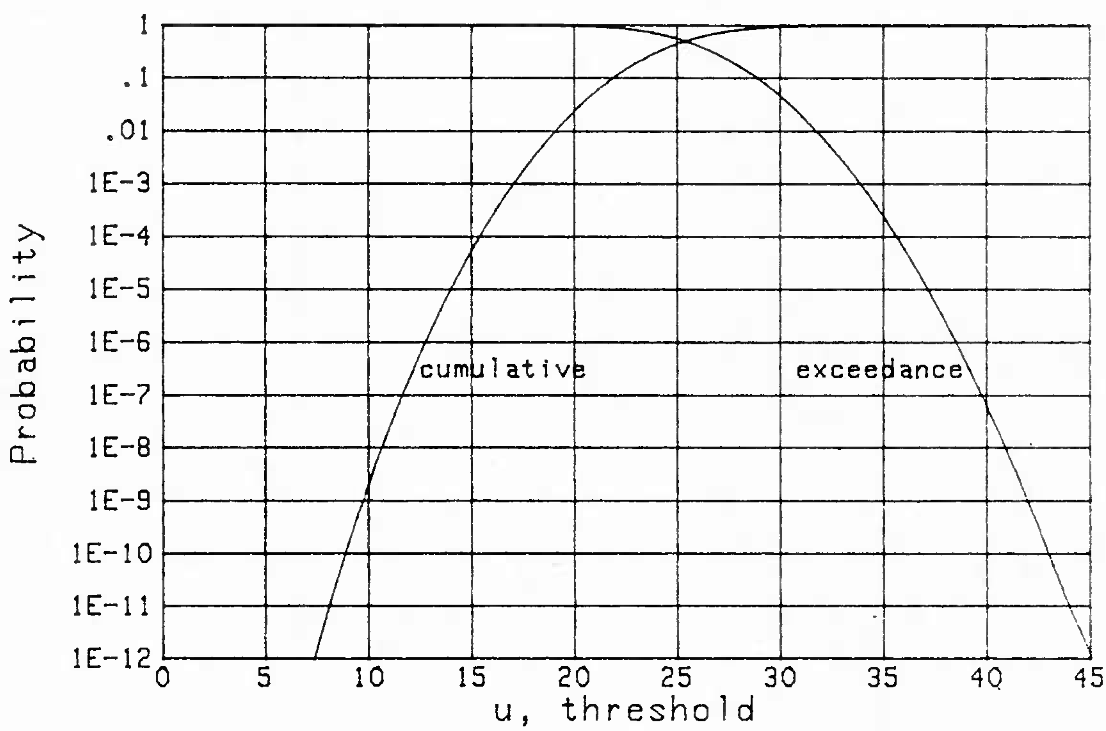

Figure 18. Distributions for Example H

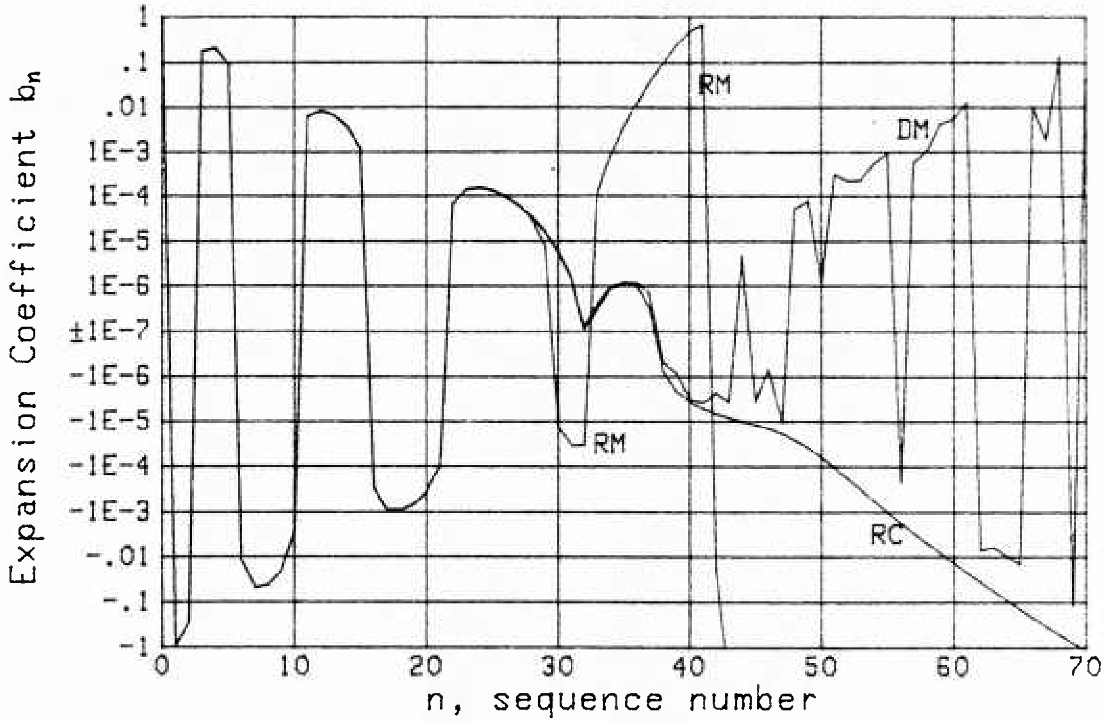

Figure 19. Generalized Laguerre; Example I

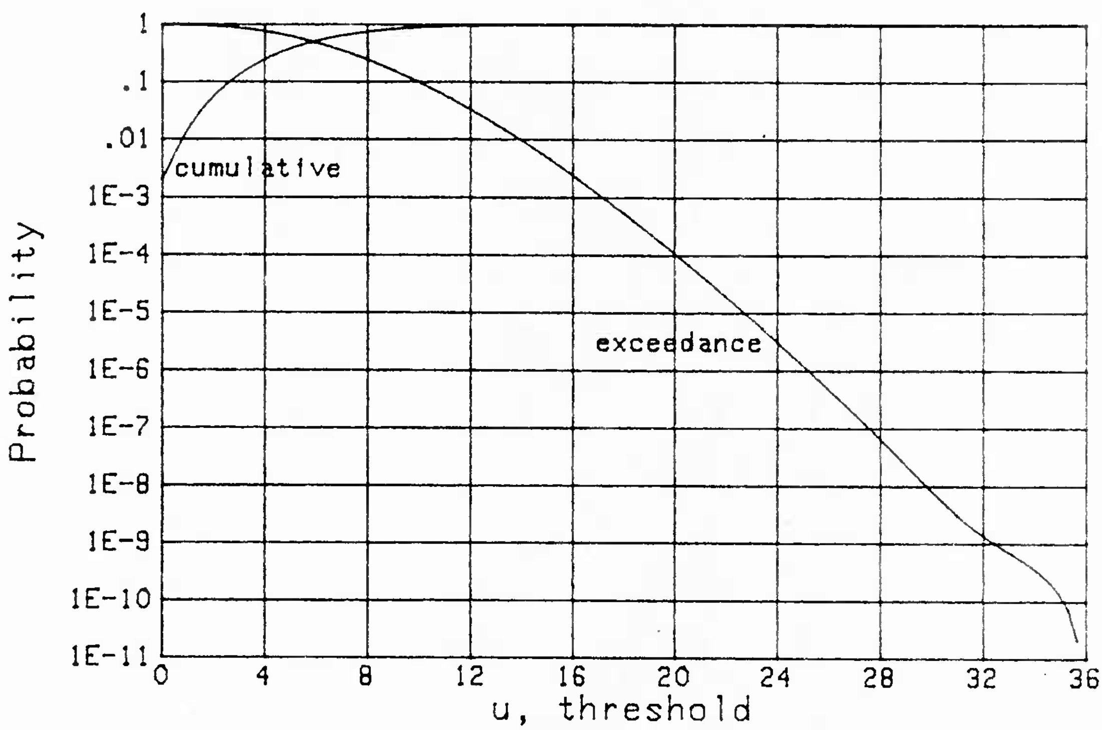

Figure 20. Distributions for Example I

contrast with the Hermite expansion coefficients in figure 7, where values in the 1E-3 range were achieved, values in the 1E-6 range can be obtained here for the generalized Laguerre expansion, for n in the mid-30s. The DM result was previously given in [4, figure D-1].

The distributions for N = 32 terms of the RC procedure are plotted in figure 20. This result is considerably better than the Hermite expansion in figure 8; instead of the wiggles which developed at 1E-4 in figure 8, the curve in figure 20 is smooth down to the 1E-8 probability level, and then develops a bump. Also, the cumulative distribution function is accurate at $u = 0$, where it takes on the value $P_0 = .002$ given in (150), and is zero for $u < 0$. This cumulative distribution function was previously given in [4, figure 8].

The probability density function for N = 32 terms of the RC procedure is given in figure 21; this result was previously given in [4, figure 9]. It is significantly better near the origin than the Hermite approximation given earlier in figure 9, which developed negative values for $u < 0$. In order to see what the probability density function does for larger u values, the same probability density function is plotted on a logarithmic ordinate in figure 22. It is accurate to the 1E-9 level but then develops a hook that is incorrect; however, this approximation remains positive even at this very low value of the density, whereas the corresponding result via a Hermite expansion in figure 10 developed negative values. The estimated errors in figures 20 and 22 are evaluated in a later section.

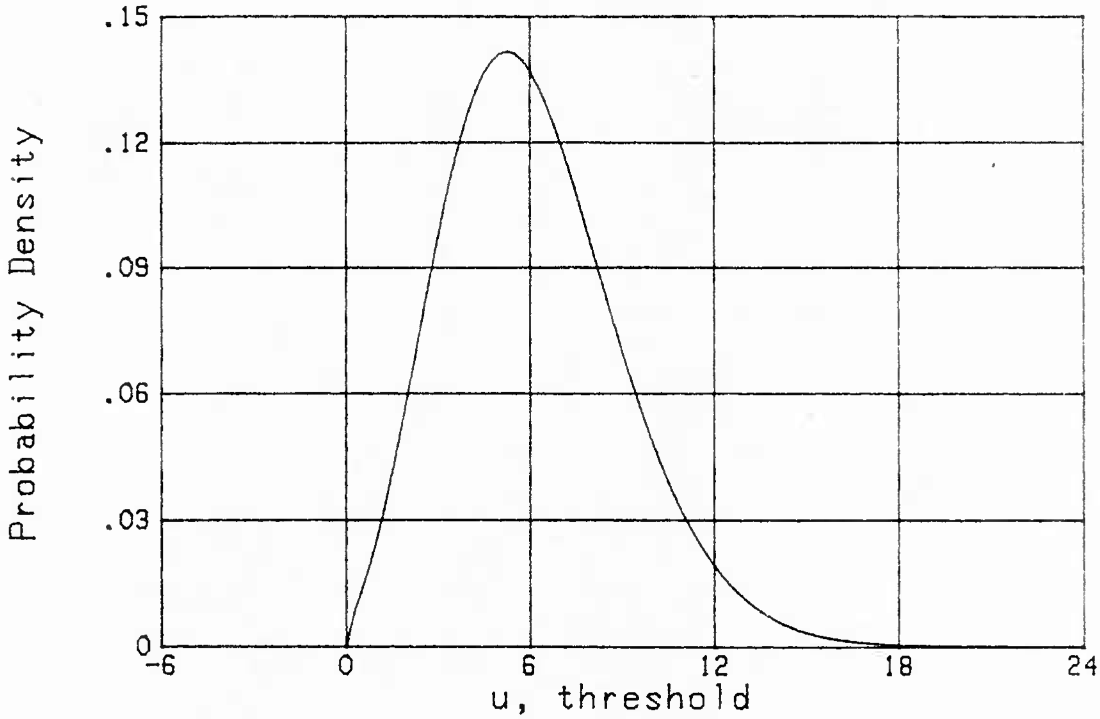

Figure 21. Linear Density for Example I

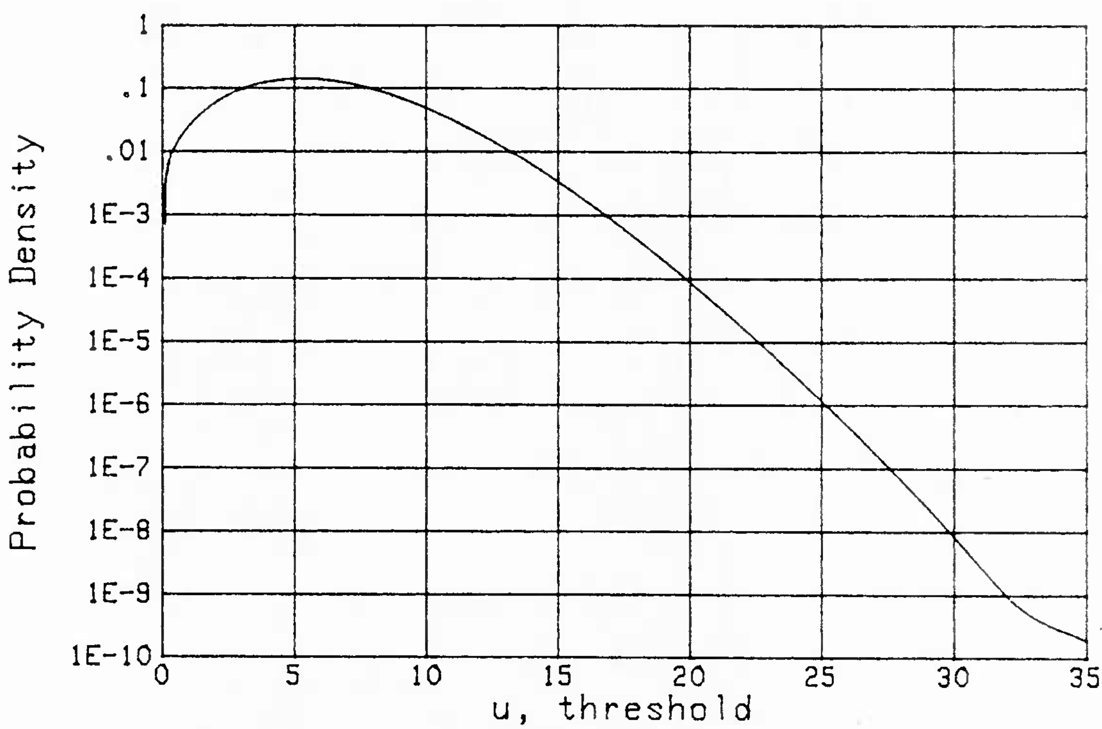

Figure 22. Log Density for Example I

EXAMPLE J

This last example is for probability density function

$$p(u) = \frac{1}{2} \exp\left(-u^{1/2}\right) \quad \text{for } u > 0 , \tag{172}$$

for which the moments are

$$\mu_n = (2n+1)! \quad . \tag{173}$$

The characteristic function and cumulants are not available in any convenient analytic form.

This is a particularly difficult example, since the characteristic function expansion in (6) has a zero radius of convergence; thus the moments do not uniquely determine the probability density function or cumulative distribution function. Also, the error integral in (21) is always infinite; in fact, regardless of the choice of weighting parameters α and β used in the generalized Laguerre series, the expansion coefficients $\{b_n\}$ always diverged. Nevertheless, a search of parameter values led to a pair of selections, namely,

$$\alpha = -.35, \qquad \beta = 30, \tag{174}$$

for which the expansion coefficients had an initial decay to the 1E-2 level before divergence took over; see figure 23. In fact, the identical same results were obtained for all three methods, RC, DM, RM; this is probably due to the fact that divergence of $\{b_n\}$ dominated before round-off error became significant.

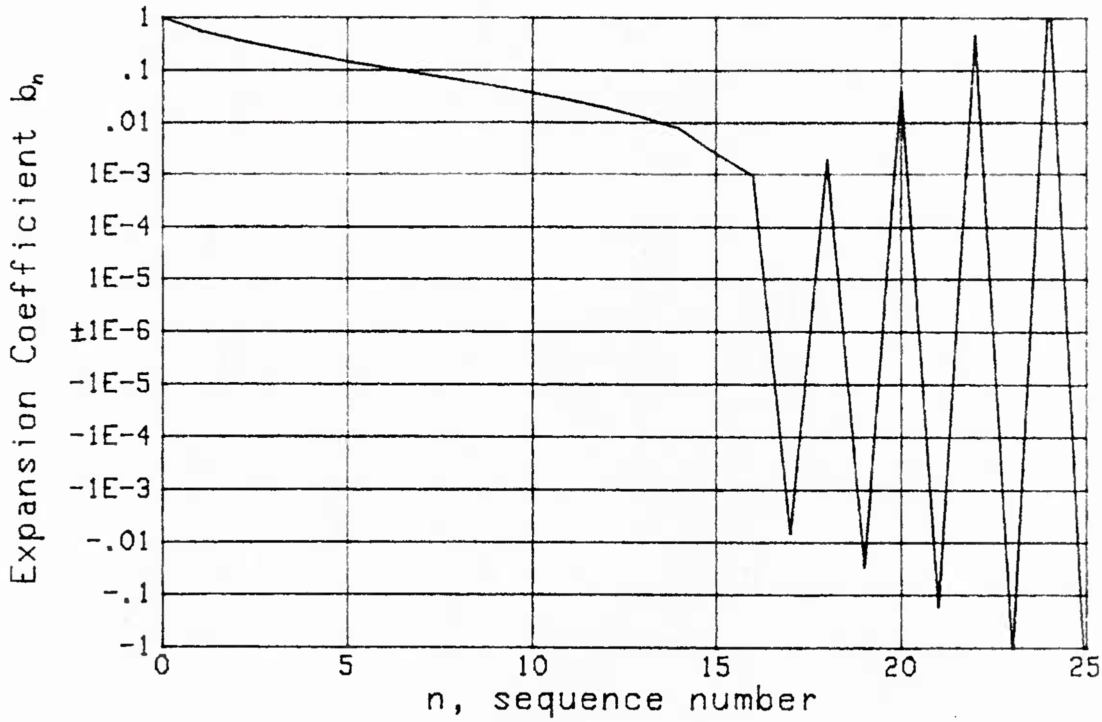

Figure 23. Generalized Laguerre; Example J

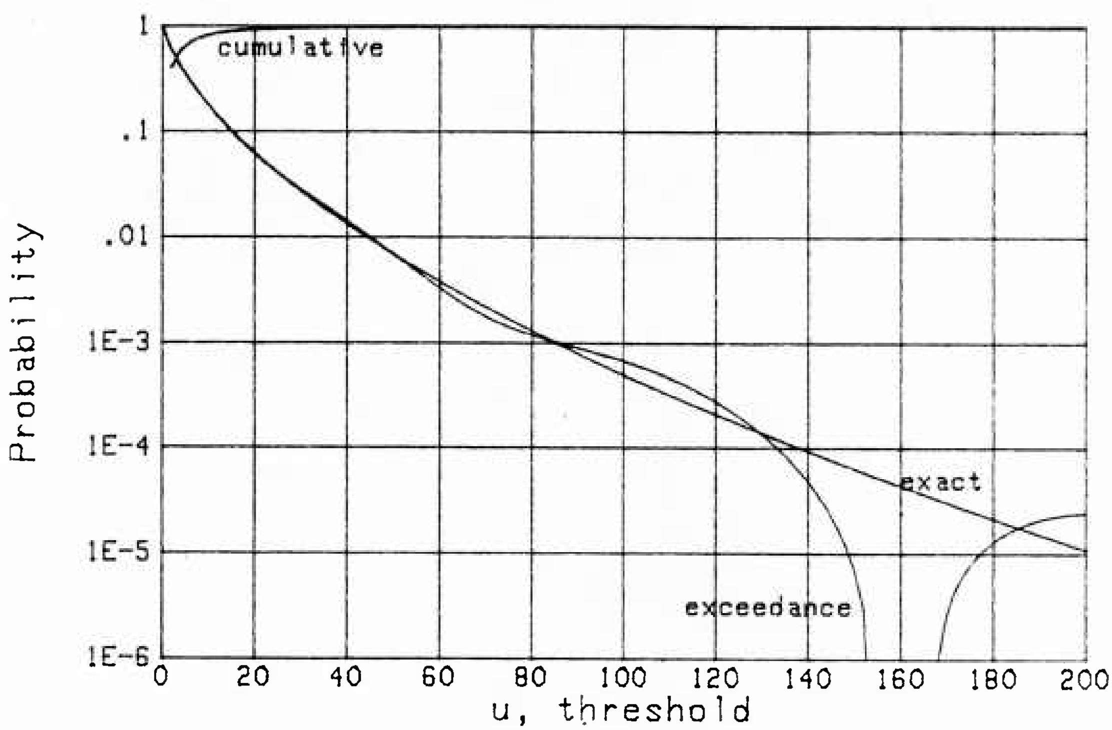

Figure 24. Distributions for Example J

The distributions are plotted in figure 24 for N = 15 terms of the generalized Laguerre series. Comparison with the exact exceedance distribution function

$$1 - P(u) = (1 + u^{1/2}) \exp(-u^{1/2}) \quad \text{for } u > 0 \tag{175}$$

reveals that the approximation is decent down to the .01 probability level, but then oscillates more and more violently as u increases. Thus even in this non-unique example, a limited-quality approximation is achieved by the generalized Laguerre series; this example confirms the comment in [3, p. 167] that, even for a divergent series, a limited number of expansion coefficients often gives a satisfactory approximation.

The exact and approximate probability density functions are plotted on a linear ordinate in figure 25, and on a logarithmic ordinate in figure 26, using N = 15 terms of the generalized Laguerre series, when the expansion coefficients were determined by the DM method. The approximate probability density function is negative for $150 < u < 190$, around the 1E-6 level. The estimated errors of the approximations in figures 24 and 26 will be developed in the next section.

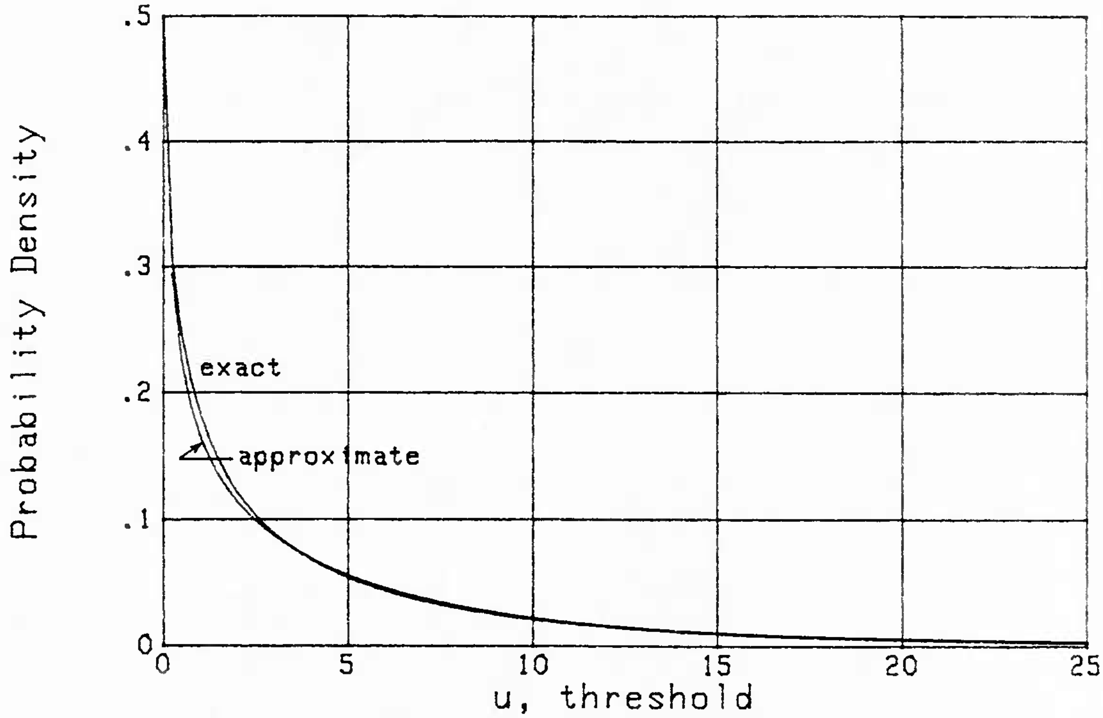

Figure 25. Linear Density for Example J

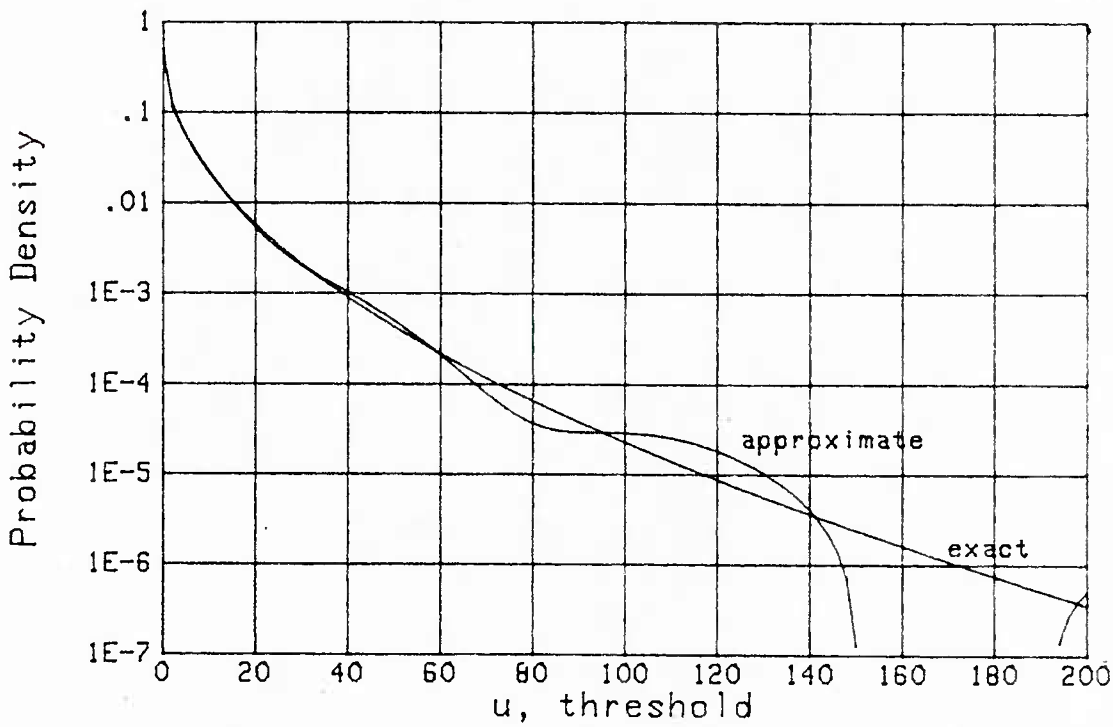

Figure 26. Log Density for Example J

ESTIMATED ERRORS OF APPROXIMATIONS

When the calculations of the approximate cumulative or exceedance distribution functions or the corresponding probability density function are made, it would be very useful to have a rough estimate of their reliability. One way, as discussed in the previous sections, is to look for nonsmooth or anomalous behavior on the tails of the functions. Here, we will develop a more quantitative estimate of the error and superpose it on some of the previous examples, for confirmation.

Both the Hermite and generalized Laguerre orthonormal polynomials oscillate with n and decay slowly. The same general behavior is true of expansion coefficients $\{b_n\}$. This leads to summations for the various functions with terms that also oscillate and decay. A rough estimate of the error is afforded by the envelope of these oscillations, evaluated at the first neglected term of the summation. This procedure will be pursued for both types of expansions; how useful it is will be indicated by numerical examples.

HERMITE EXPANSION

The following result for the envelope of the Hermite polynomial is obtained from [5, 6.1.39 and 22.5.18] and [7, 8.22.8]:

$$\mathrm{Env}\left\{(n!)^{-1/2}\ He_n(x)\right\} \sim \exp(x^2/4)\left(\frac{2}{\pi n}\right)^{1/4} \quad \text{as } n \to +\infty . \tag{176}$$

Also, from (46) and (47), the n-th term of the approximate probability density function is

$$\frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right)\,b_n\,(n!)^{-1/2}\,He_n\left(\frac{u-\alpha}{\beta}\right)\,. \tag{177}$$

Then the magnitude of the error of the probability density function approximation, if the n-th term is the first one neglected, is roughly

$$E_n\,(u;p) \equiv \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right)\,\mathrm{Env}\{b_n\}\,\mathrm{Env}\left\{(n!)^{-1/2}\,He_n\left(\frac{u-\alpha}{\beta}\right)\right\} =$$

$$\sim \left[2^{1/4}\,\pi^{3/4}\,\beta\right]^{-1}\,\exp\left(-\frac{(u-\alpha)^2}{4\beta^2}\right)\,n^{-1/4}\,\mathrm{Env}\{b_n\}\ \text{as}\ n \to +\infty. \tag{178}$$

Here we used (176).

As for the cumulative distribution function, we have from (47)-(49), the n-th term of the approximation as

$$-\phi\left(\frac{u-\alpha}{\beta}\right)\,b_n\,(n!)^{-1/2}\,He_{n-1}\left(\frac{u-\alpha}{\beta}\right)\,. \tag{179}$$

The magnitude of the error for the cumulative and exceedance distribution functions, if the n-th term is the first one neglected, is then defined as

$$E_n(u;P) \equiv \phi\left(\frac{u-\alpha}{\beta}\right)\,\mathrm{Env}\,\{b_n\}\,\mathrm{Env}\left\{(n!)^{-1/2}\,He_{n-1}\left(\frac{u-\alpha}{\beta}\right)\right\} =$$

$$\sim \left[2^{1/4}\,\pi^{3/4}\right]^{-1}\exp\left(-\frac{(u-\alpha)^2}{4\beta^2}\right)\,n^{-3/4}\,\mathrm{Env}\,\{b_n\}\ \text{as}\ n \to +\infty. \tag{180}$$

Again, (176) was of crucial importance in getting this result.

Since the above estimates are asymptotic in n, they will be most reliable for n large; their use for small n could be very misleading. The way to use these error estimates for the density and distribution approximations is as follows. First, a search on α and β, to find the fastest decaying expansion coefficients $\{b_n\}$, is conducted. The weighting parameter values, α and β, and the corresponding envelope value of the expansion coefficients $\{b_n\}$ at the point, n, where round-off error becomes dominant, are then noted. (For example, for figure 7, we observe that $\mathrm{Env}\{b_n\} \approx 2E-3$ at $n = 65$, when $\alpha = 6.1$, $\beta = 4.3$; see example D.) Then (178) and (180) can be computed and plotted in the ranges of u of interest.

An example of this procedure for the shot noise process in example D is given in figures 27 and 28. In particular, the approximate results are repeated from figures 8 and 10, and error measures (180) and (178), respectively, are superposed as dashed lines, each on the appropriate figure. Just where the approximations develop large wiggles, the errors are of comparable magnitude, indicating unreliable estimates there.

It should be observed from these figures (or from (178) and (180)) that the absolute error is maximum at $u = \alpha$, but that the relative error is a minimum in that neighborhood. Also, although the absolute error decays with u, the correct answer decays faster, leading to an increasing relative error, which eventually becomes so excessive in the tails of the various functions that the approximations are useless.

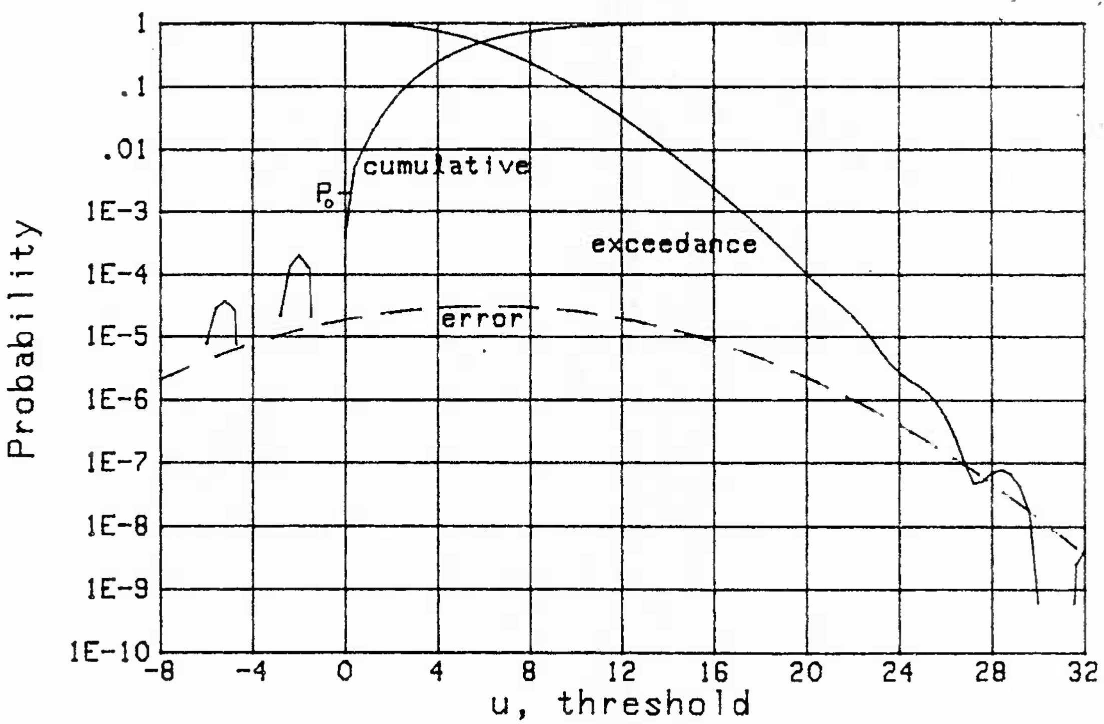

Figure 27. Estimated Error of Figure 8

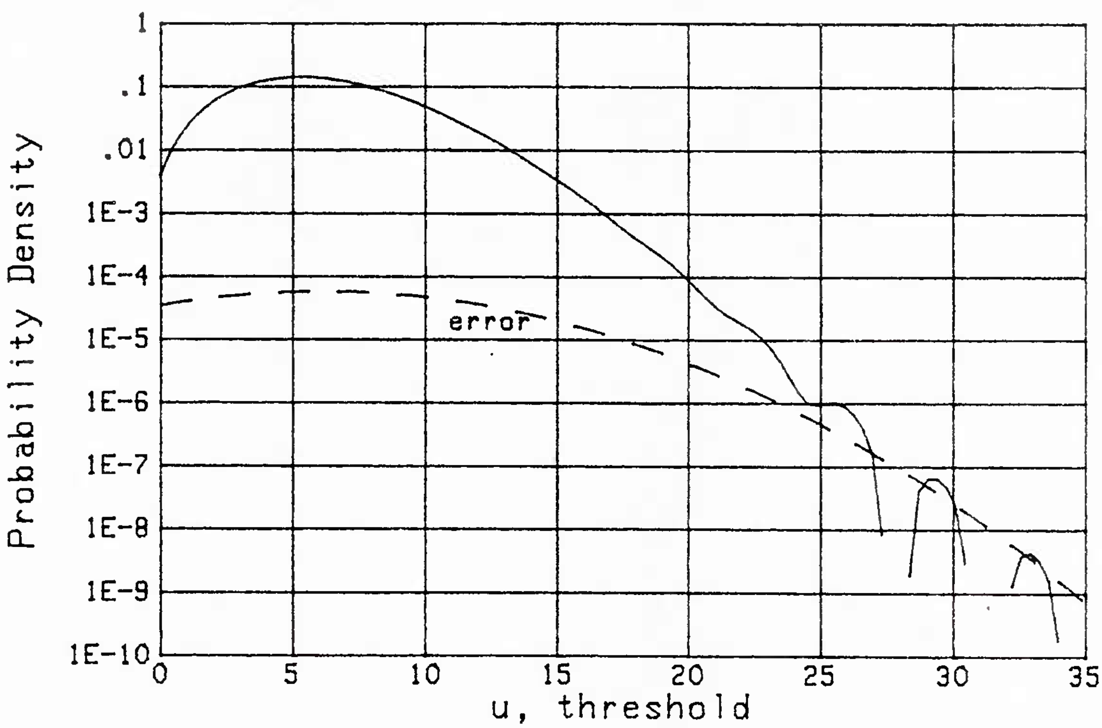

Figure 28. Estimated Error of Figure 10

GENERALIZED LAGUERRE EXPANSION

The details for the generalized Laguerre series are very similar to those above and so will be abbreviated. The envelope of the generalized Laguerre polynomial is [7, 8.22.1]

$$\mathrm{Env}\left\{L_n^{(\alpha)}(x)\right\} \sim \pi^{-\frac{1}{2}}\, e^{\frac{x}{2}}\, x^{-\frac{\alpha}{2}-\frac{1}{4}}\, n^{\frac{\alpha}{2}-\frac{1}{4}} \quad \text{as } n \to +\infty, \quad \text{for } x > 0\,. \tag{181}$$

From (91) and (92), the n-th term of the approximate probability density function is

$$\frac{u^{\alpha}\exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)}\; b_n \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_n^{(\alpha)}\left(\frac{u}{\beta}\right). \tag{182}$$

Then the magnitude of the error of the probability density function approximation is, for $u > 0$,

$$E_n(u;p) \equiv \frac{u^{\alpha}\exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)}\; \mathrm{Env}\left\{b_n\right\}\, \mathrm{Env}\left\{\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_n^{(\alpha)}\left(\frac{u}{\beta}\right)\right\} =$$

$$\sim \left[\pi\,\Gamma(\alpha+1)\,\beta^2\right]^{-1/2} \left(\frac{u}{\beta}\right)^{\frac{\alpha}{2}-\frac{1}{4}} \exp\left(-\frac{u}{2\beta}\right) n^{-1/4}\, \mathrm{Env}\left\{b_n\right\} \text{as } n \to +\infty, \tag{183}$$

where we used [5, 6.1.47] and (181). This quantity peaks at $u = \beta(\alpha-\frac{1}{2})$.

With regards to the cumulative distribution function, the n-th term of the approximation is, from (95) and (92),

$$\frac{u^{\alpha+1}\exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)} \; b_n \frac{1}{n}\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right). \tag{184}$$

Then the magnitude of the distribution error, for both the cumulative and the exceedance distribution functions, is roughly

$$E_n(u;P) \equiv \frac{u^{\alpha+1}\exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)} \; \mathrm{Env}\left\{b_n\right\} \; \mathrm{Env}\left\{\frac{1}{n}\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right)\right\} =$$

$$\sim \left[\pi\Gamma(\alpha+1)\right]^{-\frac{1}{2}} \left(\frac{u}{\beta}\right)^{\frac{\alpha}{2}+\frac{1}{4}} \exp\left(-\frac{u}{2\beta}\right) n^{-\frac{3}{4}} \; \mathrm{Env}\left\{b_n\right\} \quad \text{as } n \to +\infty, \text{ for } u > 0, \tag{185}$$

upon use of (181). This quantity reaches its peak at $u = \beta(\alpha+\frac{1}{2})$.

An application of these results to the shot noise process, which was re-investigated in example I via the generalized Laguerre series, is given in figures 29 and 30. Specifically, the approximate results from figures 20 and 22 have been repeated, and error measures (185) and (183), respectively, superposed as dashed lines. They confirm the earlier observations that the distribution and density approximations are reliable until the anomalous behavior on the tails manifests itself.

The difficult example J is considered in figures 31 and 32. Since the expansion coefficient sequence $\{b_n\}$ in figure 23 diverged for large n, the selection of n = 15, as used in figures 24–26, is not the large value needed to justify the use of (183) and (185). Thus, the dashed curves on figures 31 and 32 must be considered only as ball-park estimates; in general, the approximate error appears to be too conservative in these two figures.

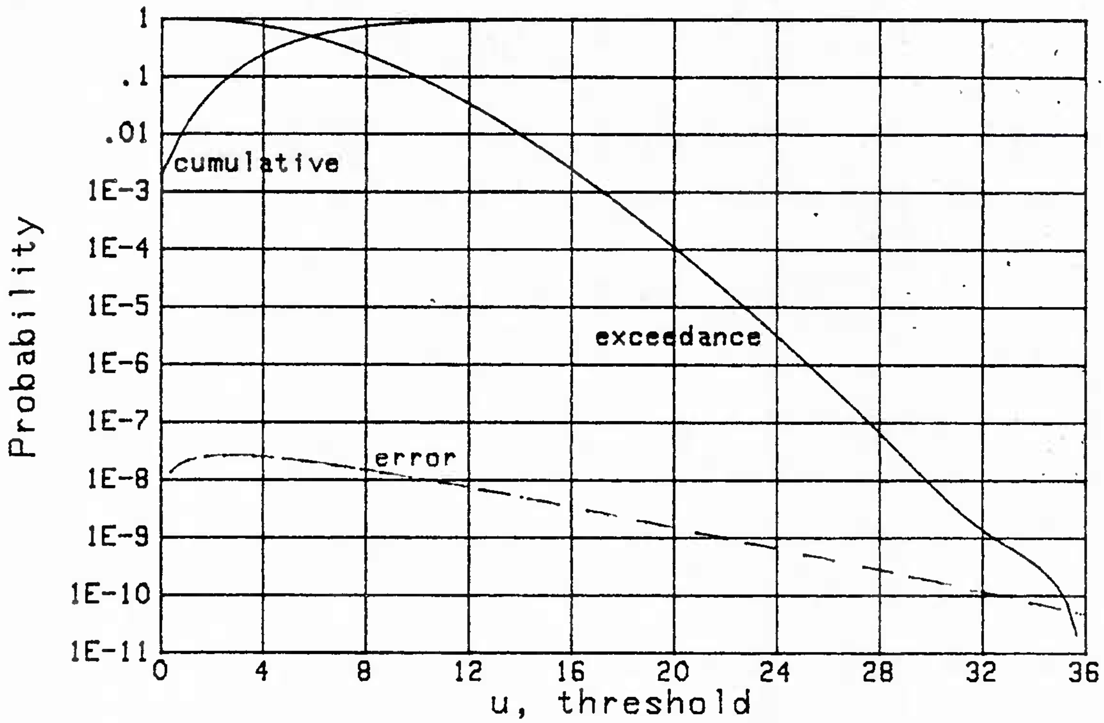

Figure 29. Estimated Error of Figure 20

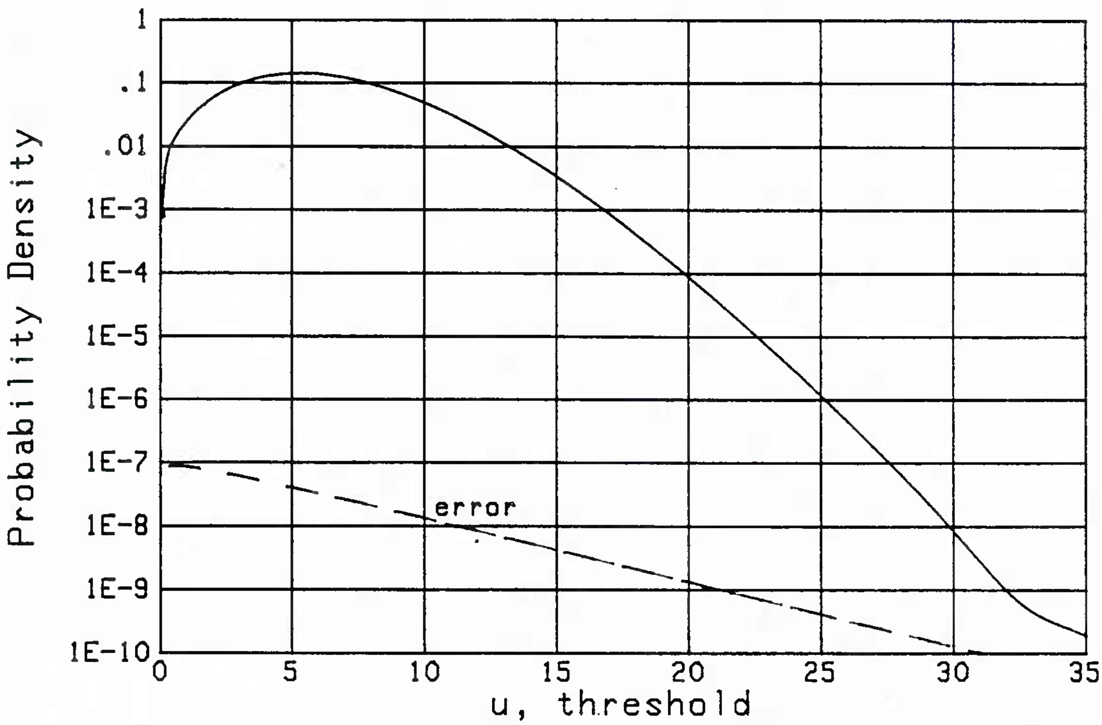

Figure 30. Estimated Error of Figure 22

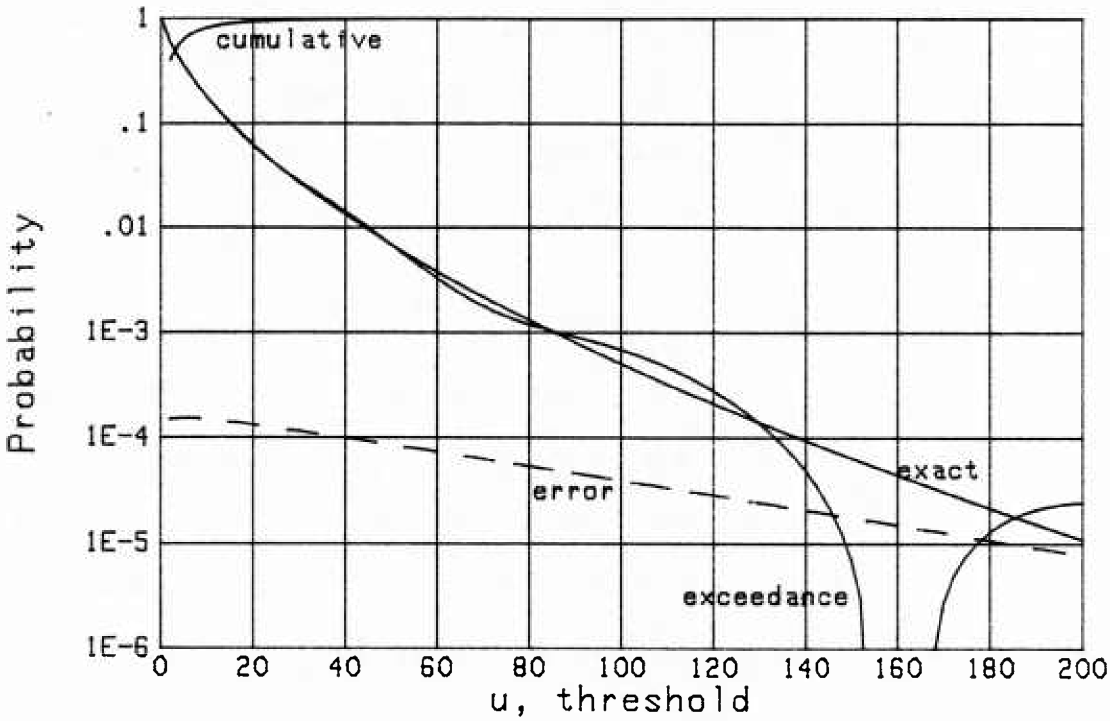

Figure 31. Estimated Error of Figure 24

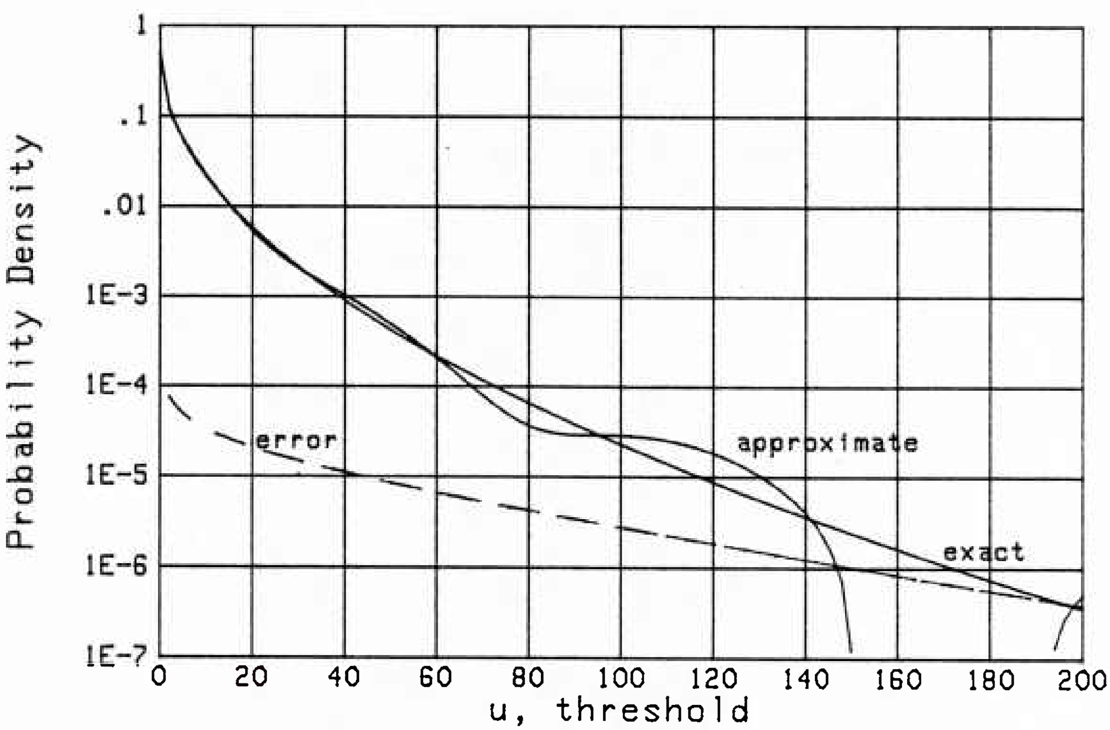

Figure 32. Estimated Error of Figure 26

Finally, the Rice variate of example G is re-considered in figure 33. We took $\mathrm{Env}\{b_n\} \approx 3\mathrm{E}{-4}$ at $n = 65$, by extrapolating in figure 15 from smaller n, since round-off error is becoming significant by this value. It verifies the unreliability of the approximation in figure 33 for $u > 7$.

Although all the examples in this report have the capability of evaluating either the moments or the cumulants via recursion, this is by no means necessary. Any method whatsoever of accurately calculating the starting statistics, be they moments or cumulants, is acceptable. For example, if a random variable with known probability density function q is passed through a complicated nonlinearity g, the moments of the output are given by

$$\mu_n = \int du\, g^n(u)\, q(u) \ . \tag{186}$$

These quantities could be evaluated for $0 \le n \le N$ by brute-force numerical procedures if necessary. The limit value N will depend on the accuracy with which g and q can be evaluated; if $g(u) \ge 0$ for all u, these integrals can be accomplished to a high degree of accuracy, thereby allowing large values of N to be employed.

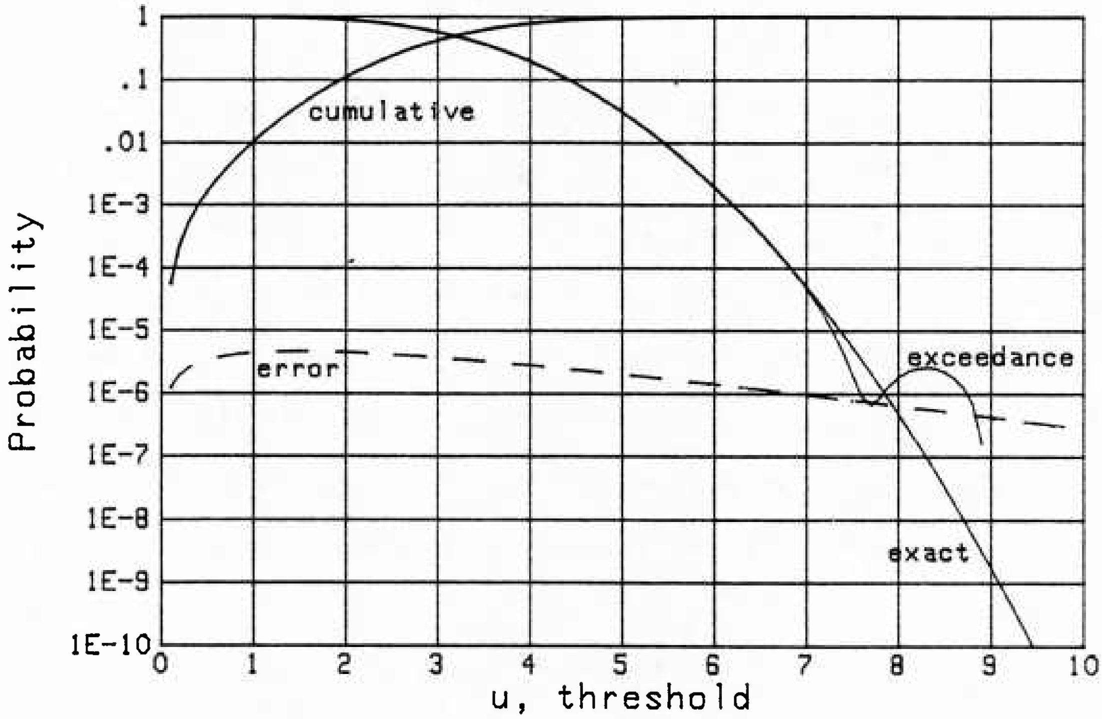

Figure 33. Estimated Error of Figure 16

DISCUSSION

Several alternative methods have been presented for obtaining either Hermite or generalized Laguerre series expansions of probability density functions or cumulative and exceedance distribution functions, by means of recursive relations involving either moments or cumulants. Furthermore, estimates of the errors of the approximations are furnished so that the reliability can be assessed. Comparisons between approximations obtained by either the Hermite or the generalized Laguerre series afford an assessment of the accuracy of each; also, the availability of three alternative recursive procedures for the expansion coefficients allows for selection of the best method and results, and determination of the amount of round-off error.

The key feature to this approach is the rapid calculation and observation of the orthonormal expansion coefficients $\{b_n\}$ for each particular guess of weighting parameters α and β. A trial and error procedure is suggested for determining α and β values that yield the set of fastest-decaying expansion coefficients. From observation of the expansion coefficients, the number of terms to retain in the series expansions is ascertained, being sure to avoid the effects of round-off error which dominates the calculated expansion coefficients $\{b_n\}$ for large n. Since the amount and location of round-off error on the plot of expansion coefficients also depends on α and β, a judicious search may be required to find acceptable weighting parameter values. Of course, a computer with a larger number of significant digits would greatly alleviate this drawback; the particular computer used for all the calculations reported here is the Hewlett-Packard 9000 Model 520 which devotes 52 bits (15.65 decimal digits) to the mantissa and 11 bits to the exponent. Failure of the technique is indicated by divergence of the expansion coefficient sequence $\{b_n\}$.

Programs for the shot noise process considered in examples D and I are presented in appendix F. Times of execution are as follows. For the Hermite series, the 80 cumulants or 80 moments required as input for figure 7 took .7 or .35 seconds, respectively. The calculation, plotting, and display of the 80 expansion coefficients in figure 7 took 1.6 seconds via the RC approach and 1.75 seconds via the two moment approaches. The computation and display of the 100-point plots of the cumulative distribution function in figure 8 and the probability density function in figure 9, each using 65 terms in the series expansion, took 1.1 and .95 seconds, respectively.

For the generalized Laguerre series, the 70 cumulants or 70 moments required as input for figure 19 took .54 seconds or .28 seconds, respectively. The calculation and display of the 70 expansion coefficients in figure 19 took 1.8 seconds via the RC approach and 1.5 seconds via the two moment approaches. The computation and display of the 100-point plots of the cumulative distribution function in figure 20 and the probability density function in figure 21 took 1.1 and .7 seconds, respectively. These execution times are short enough to allow a human observer to conduct a rapid trial-and-error search of α,β space, determine adequate parameter values, and assess their accuracy.

Alternative exact procedures for determination of cumulative and exceedance distribution functions via characteristic functions have been presented in [9, 10, 11]. Those methods generally have the potential for greater accuracy, are less subject to round-off-error, and would be preferred if possible. However, analysis of systems with nonlinearities and memory sometimes precludes or greatly hinders their application; in such cases, the current approach is a very good candidate for consideration.

The two weightings in (1) and (2), namely the Hermite and generalized Laguerre, have been investigated rather intensively here, because so many properties and recursions are available for the corresponding (orthonormal) polynomials. These properties have been utilized to derive simple recursive relations for the expansion coefficients and density and distribution functions, thereby realizing quick efficient procedures for numerical evaluation and observation.

It would be extremely useful to be able to extend these results to the weighting

$$u^{\alpha} \exp(-u^2/\beta^2) \text{ for } u > 0, \tag{187}$$

since this class of probability density functions is often encountered in nonlinear systems with Gaussian inputs. However, there are several pivotal recursive relations for the corresponding orthonormal polynomials that would be needed, and it is questionable if a fast procedure could be devised without them. Also, it is unknown if recursive procedures for the expansion coefficients in terms of moments or cumulants could be derived, as was done here for the Hermite and generalized Laguerre weightings. This is a topic worthy of further investigation.

APPENDIX A. COEFFICIENT RECURSION FOR EXPONENTIAL OF POWER SERIES

Suppose power series $\sum_{n=0}^{\infty} h_n z^n$ converges for some $|z| > 0$, and we exponentiate it, getting a new power series

$$\sum_{n=0}^{\infty} g_n z^n = \exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\} . \tag{A-1}$$

Then the lowest order coefficient is

$$g_0 = \exp(h_0) , \tag{A-2}$$

while for $k \geq 1$, we have

$$g_k = \frac{1}{k!}\left(\frac{d}{dz}\right)^k \left[\exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\}\right]_{z=0} =$$

$$= \frac{1}{k!}\left(\frac{d}{dz}\right)^{k-1}\left[\sum_{n=1}^{\infty} n\, h_n z^{n-1} \exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\}\right]_{z=0} =$$

$$= \frac{1}{k!}\sum_{p=0}^{k-1}\binom{k-1}{p}\left(\frac{d}{dz}\right)^p\left[\sum_{n=1}^{\infty} n\, h_n z^{n-1}\right]_{z=0}\left(\frac{d}{dz}\right)^{k-1-p}\left[\exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\}\right]_{z=0} =$$

$$= \frac{1}{k!}\sum_{p=0}^{k-1}\binom{k-1}{p}(p+1)!\, h_{p+1}\,(k-1-p)!\, g_{k-1-p} =$$

$$= \frac{1}{k}\sum_{p=0}^{k-1}(p+1)\, h_{p+1}\, g_{k-1-p} = \frac{1}{k}\sum_{m=1}^{k} m\, h_m\, g_{k-m} . \tag{A-3}$$

Thus we have the recursion for coefficients $\{g_k\}$ in terms of the $\{h_m\}$:

$$g_k = \frac{1}{k}\sum_{m=1}^{k} m\, h_m\, g_{k-m} \quad \text{for } k \geq 1, \quad g_0 = \exp(h_0) . \tag{A-4}$$

If we now refer to (6) and (7) and identify

$$g_n = \mu_n / n! \,, \qquad h_n = \chi_n / n! \,, \tag{A-5}$$

there follows the moments in terms of the cumulants according to

$$\mu_k = \sum_{m=0}^{k-1} \binom{k-1}{m} \chi_{k-m}\, \mu_m \quad \text{for } k \geq 1, \qquad \mu_0 = \exp(\chi_0) \;. \tag{A-6}$$

This is a slight generalization of [6, (10)]. This equation is immediately inverted, to yield the cumulants in terms of moments:

$$\chi_k = \frac{1}{\mu_0}\left[\mu_k - \sum_{m=1}^{k-1} \binom{k-1}{m} \chi_{k-m}\, \mu_m\right] \quad \text{for } k \geq 1, \qquad \chi_0 = \ln \mu_0 \,, \tag{A-7}$$

which generalizes [6, (11)].

In terms of the normalized cumulants and moments defined in (62) and (69) respectively, we have

$$\hat{\mu}_k = \frac{1}{k} \sum_{m=0}^{k-1} \hat{\chi}_{k-m}\, \hat{\mu}_m \quad \text{for } k \geq 1, \qquad \hat{\mu}_0 = \exp(\chi_0) \,, \tag{A-8}$$

and

$$\hat{\chi}_k = \frac{1}{\hat{\mu}_0}\left[k\, \hat{\mu}_k - \sum_{m=1}^{k-1} \hat{\chi}_{k-m}\, \hat{\mu}_m\right] \quad \text{for } k \geq 1, \qquad \left(\chi_0 = \ln \mu_0\right) . \tag{A-9}$$

APPENDIX B. EXPANSION OF $He_n(x+y)$

The quantity $He_n(x+y)$ is a polynomial of degree n in y. Therefore we can expand

$$He_n(x+y) = \sum_{m=0}^{n} \gamma_m \frac{y^m}{m!} , \tag{B-1}$$

where γ_m will also depend on n and x. In fact,

$$\gamma_m = \left(\frac{\partial}{\partial y}\right)^m \left[He_n(x+y)\right]_{y=0} = \left(\frac{\partial}{\partial t}\right)^m \left[He_n(t)\right]_{t=x} =$$

$$= \left(\frac{\partial}{\partial t}\right)^{m-1} \left[n\ He_{n-1}(t)\right]_{t=x} = n(n-1)\ \ldots\ (n-m+1)\ He_{n-m}(x) =$$

$$= \frac{n!}{(n-m)!} He_{n-m}(x) , \tag{B-2}$$

where we used [5, 22.8.8] repeatedly. Using (B-2) in (B-1), we have the alternative forms for the expansion,

$$He_n(x+y) = \sum_{m=0}^{n} \binom{n}{m} He_{n-m}(x)\ y^m =$$

$$= \sum_{m=0}^{n} \binom{n}{m} He_{n-m}(y)\ x^m =$$

$$= \sum_{k=0}^{n} \binom{n}{k} He_k(y)\ x^{n-k} \quad \text{for } n \geq 0 . \tag{B-3}$$

APPENDIX C. EVALUATION OF $I_n(y)$ IN (94)

We have, from (94),

$$I_n(y) = \int_0^y dx\ x^\alpha\ e^{-x}\ L_n^{(\alpha)}(x) \quad \text{for } n \geq 0\ . \tag{C-1}$$

Then

$$I_0(y) = \int_0^y dx\ x^\alpha\ e^{-x}\ 1 = \gamma(\alpha+1,y) = \frac{y^{\alpha+1}\ e^{-y}}{\alpha+1}\ {}_1F_1(1;\alpha+2;y), \tag{C-2}$$

using [5, 22.4.7, 6.5.2, and 6.5.12]. Also, we have from [5, 22.11.6],

$$x^\alpha\ e^{-x}\ L_n^{(\alpha)}(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left\{e^{-x}\ x^{\alpha+n}\right\}\ . \tag{C-3}$$

Then for $n \geq 1$, (C-1) can be developed as

$$I_n(y) = \int_0^y dx\ \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left\{e^{-x}\ x^{\alpha+n}\right\} =$$

$$= \frac{1}{n!}\int_0^y d\left(\frac{d}{dx}\right)^{n-1}\left\{e^{-x}\ x^{\alpha+n}\right\} = \frac{1}{n!}\left(\frac{d}{dy}\right)^{n-1}\left\{e^{-y}\ y^{\alpha+n}\right\} =$$

$$= \frac{1}{n}\ y^{\alpha+1}\ e^{-y}\ L_{n-1}^{(\alpha+1)}(y)\ , \tag{C-4}$$

where we set the lower limit of the evaluated integral to zero since $\alpha+n \geq \alpha+1 > 0$.

APPENDIX D. FOURIER TRANSFORM OF GENERALIZED LAGUERRE POLYNOMIAL

We wish to evaluate transform

$$A(\omega) = \int_0^\infty dt\ e^{i\omega t}\ t^\alpha\ e^{-t}\ L_n^{(\alpha)}(t)\ . \tag{D-1}$$

Now

$$n!\ t^\alpha\ e^{-t}\ L_n^{(\alpha)}(t) = \left(\frac{d}{dt}\right)^n \left\{e^{-t}\ t^{\alpha+n}\right\} \quad \text{for } n \geq 0\ , \tag{D-2}$$

according to [5, 22.11.6]. Therefore for $n \geq 1$,

$$\begin{aligned} n!\ A(\omega) &= \int_0^\infty dt\ e^{i\omega t} \left(\frac{d}{dt}\right)^n \left\{e^{-t}\ t^{\alpha+n}\right\} = \\ &= \int_0^\infty e^{i\omega t}\ d\left(\frac{d}{dt}\right)^{n-1} \left\{e^{-t}\ t^{\alpha+n}\right\} = \\ &= -i\omega \int_0^\infty dt\ e^{i\omega t} \left(\frac{d}{dt}\right)^{n-1} \left\{e^{-t}\ t^{\alpha+n}\right\}\ , \end{aligned} \tag{D-3}$$

where we used integration by parts with the fact that the integrated part is zero at $t = 0$ and ∞, since $\alpha+n \geq \alpha+1 > 0$. Repeated integration by parts then yields

$$n!\ A(\omega) = (-i\omega)^n \int_0^\infty dt\ e^{i\omega t}\ e^{-t}\ t^{\alpha+n} = \Gamma(\alpha+1+n)\ \frac{(-i\omega)^n}{(1-i\omega)^{\alpha+1+n}}\ . \tag{D-4}$$

This is the result quoted in (104).

APPENDIX E. RECURRENCE FOR EXAMPLE C

The starting point is the moment expression in (142):

$$\mu_n = \Lambda \frac{\theta^{\zeta}\,\Gamma\left(\frac{n}{2}+h\right)\ \omega^{n+2h}}{2^{\zeta+1}\,\Gamma(\zeta+1)}\ {}_1F_1\left(\frac{n}{2}+h;\zeta+1;z\right) , \tag{E-1}$$

where $h = (\gamma+\zeta+1)/2$, $z = \omega^2\theta^2/4$. Denote the ${}_1F_1$ term in (E-1) by F_n, and the leading factor by G_n; thus $\mu_n = G_n\ F_n$. There follows immediately

$$G_n = G_{n-2}\ \omega^2\left(\frac{n}{2}+h-1\right) \quad \text{for } n \geq 2 . \tag{E-2}$$

For the ${}_1F_1$ function, we refer to [5, 13.4.1] to get

$$F_n = \frac{1}{\frac{n}{2}+h-1}\left[(n+2h-3-\zeta+z)F_{n-2} + \left(\zeta+2-\frac{n}{2}-h\right)F_{n-4}\right] . \tag{E-3}$$

If we substitute (E-2) and (E-3) into $\mu_n = G_n\ F_n$, and then re-apply (E-2) in the second term, we obtain

$$\mu_n = \omega^2(n+\gamma-2+z)\ \mu_{n-2} - \frac{1}{4}\,\omega^4\left[(n+\gamma-3)^2 - \zeta^2\right]\mu_{n-4} ; \tag{E-4}$$

we also eliminated h. Starting values for μ_n can be obtained from (E-1).

For the special case (143) and (144), (E-4) reduces to

$$\mu_n = \omega^2(n-1+z)\ \mu_{n-2} - \frac{1}{4}\,\omega^4(n-2)^2\ \mu_{n-4} , \tag{E-5}$$

with starting values

$$\mu_0 = 1, \quad \mu_1 = \frac{1}{2}\,\pi^{1/2}\,\omega\, e^{-z}\ {}_1F_1\!\left(\frac{3}{2};1;z\right) ,$$

$$\mu_2 = \omega^2(1+z), \quad \mu_3 = \frac{3}{4}\,\pi^{1/2}\,\omega^3\, e^{-z}\ {}_1F_1\!\left(\frac{5}{2};1;z\right) . \tag{E-6}$$

Kummer's transformation [5, 13.1.27] was employed in this last equation; these forms afford accurate starting values for recursion (E-5).

APPENDIX F. PROGRAM LISTINGS

Eight programs are listed in this appendix. They are given in BASIC for the Hewlett Packard 9000 Model 520 computer. For ease of reference, a shorthand notation is adopted:

P	denotes	cumulative or exceedance distribution function
p	denotes	probability density function
H	denotes	Hermite expansion
L	denotes	generalized Laguerre expansion
RC	denotes	recursively via cumulants
DM	denotes	directly via moments
RM	denotes	recursively via moments

Table F-1. Shorthand Notation

Then, for example, the combination PHRC means that this program yields the cumulative or exceedance distribution function in terms of a Hermite expansion, by means of expansion coefficients determined recursively via cumulants. The eight programs listed here are, in order,

PHRC	Figures 7 and 8
pHRC	Figures 7, 9, and 10
PHDMandRM	Figures 7 (and 8)
pHDMandRM	Figures 7 (and 9, 10)
PLRC	Figures 19 and 20
pLRC	Figures 19, 21, and 22
PLDMandRM	Figures 19 (and 20)
pLDMandRM	Figures 19 (and 21, 22)

Table F-2. Program Abbreviations

The combination DMandRM means that this program gives the expansion coefficients directly via moments as well as recursively via moments; the user must select the procedure of interest.

The only input statistics we have given a listing for here is the shot noise process used in examples D and I; in particular, the cumulant and moment routines are listed at the very end of PHRC and PHDMandRM, respectively. The figure references given in table F-2 indicate where each particular program was used in this report; the parenthetical references are alternative ways of generating those figures. The remaining figures in this report require that the cumulant and moment subroutines be replaced by the appropriate statistics of interest.

To save space, no subroutines are listed more than once; instead, comments are made indicating where the needed routines are located, according to the coding in table F-2. For example, in program PHDMandRM, function subprogram FNPhi, line 570, the comment is made that this routine has already been listed in PHRC.

We now explain some of the details of the PHRC program, as an example, so that a user can apply these techniques and routines to his particular problem. The user must specify M in line 30, which is the maximum order of approximation desired, or the number of cumulants or moments that can be calculated. The notation DOUBLE in line 40 denotes INTEGER variables. The user must select α and β in lines 130,140; if they are chosen equal to α_0,β_0 which have been computed in lines 110,120, then expansion coefficients $a_1 = a_2 = 0$, or equivalently $b_1 = b_2 = 0$. However, this choice is recommended only as a starter on the search in α,β space.

The CALL in line 150 is to the subroutine which calculates the expansion coefficients for a Hermite series, recursively via cumulants, as can be deciphered from the abbreviated subroutine title. The expansion coefficients $\{b_n\}$ are calculated and the running sum of b_n^2 is calculated, both of which are printed on the CRT vs n. Also, a plot of the expansion coefficients $\{b_n\}$ is made in this subroutine, from which the user must decide on the order, N, to employ in the approximate cumulative and exceedance distribution function; alternatively, he can reject the sequence of $\{b_n\}$ so obtained, and re-run the program with different α,β values.

When a satisfactory α,β pair is obtained, the limits u_1,u_2 on the range of arguments of the distribution must also be specified; this selection is aided by the print-out of the center and rms width of the density under investigation. A plot of 100 values of the cumulative and exceedance distribution functions is then made on a logarithmic ordinate. The various subroutines are self-explanatory and are keyed to the equation numbers in this report.

PROGRAM PHRC

```
 10 !  STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u)    TR 7377, FIGURE 8
 20 !  COEFFICIENTS OF HERMITE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
 30    M=80    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
 40    DOUBLE M,I,N,K                     !  INTEGERS < 2^31 = 2,147,483,648
 50    REDIM Cum(0:M),A(0:M),He(0:M)
 60    REAL Cum(0:100),A(0:100),He(0:100),P(0:100)
 70    CALL Cumulants(M,P0,Cum(*))        !  P0 IS STEP AT ORIGIN
 80    Center=Cum(1)                      !  CENTER OF PDF pc(u)
 90    R2=Cum(2)                          !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                        !  RMS SPREAD OF pc(u)
110    Alpha0=Center                      !  THE CHOICES   Alpha=Alpha0   AND
120    Beta0=Rms                          !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=Center
140      Beta=Rms*1.5
150    CALL Coeffhr_via_cum(M,Alpha,Beta,Cum(*),A(*))     !   RC
160    PRINT "Center = ";Center
170    PRINT "Rms =";Rms
180    F1=1./SQR(2.*PI)
190      INPUT "ORDER AND LIMITS:",N,U1,U2
200    PRINT "ORDER AND LIMITS:",N;U1;U2
210    Du=(U2-U1)/100.
220    PLOTTER IS "GRAPHICS"
230    GRAPHICS ON
240    WINDOW U1,U2,-10.,0.
250    GRID Du*10.,1.
260    FOR I=0 TO 100
270    U=U1+Du*I
280    T=(U-Alpha)/Beta
290    CALL Hermite(N,T,He(*))
300    Sum=0.
310    FOR K=1 TO N
320    Sum=Sum+A(K)*He(K-1)
330    NEXT K
340    P=A(0)*FNPhi(T)-F1*EXP(-.5*T*T)*Sum     !  PROBABILITY THAT RV < U
350    IF U>=0. THEN P=P+P0                    !  ADDITION OF STEP AT ORIGIN
360    P(I)=P
370    IF P>0. THEN 400
380    PENUP
390    GOTO 410
400    PLOT U,LGT(P)
410    NEXT I
420    PENUP
430    FOR I=0 TO 100
440    U=U1+Du*I
450    P1=1.-P(I)
460    IF P1>0. THEN 490
470    PENUP
480    GOTO 500
490    PLOT U,LGT(P1)
500    NEXT I
510    PENUP
520    GOTO 190
530    END
540    !
```

PROGRAM PHRC (cont'd)

```
550     DEF FNPhi(X)           !  HART, page 140, #5708 & #5725                eq. 41
560     Y=ABS(SQR(.5)*X)
570     SELECT Y
580     CASE <8.
590     P=1631.76026875371470+Y*(456.261458706092631+Y*(86.0827622119485951+Y*
(10.0648589749095425+Y*.564189586761813614)))
600     P=3723.50798155480672+Y*(7113.66324695404987+Y*(6758.21696411048589+Y*
(4032.26701083004974+Y*P)))
610     Q=7542.47951019347576+Y*(2968.00490148230872+Y*(817.622386304544077+Y*
(153.077710750362216+Y*(17.8394984391395565+Y))))
620     Q=3723.50798155480654+Y*(11315.1920818544055+Y*(15802.5359994020425+Y*
(13349.3465612844574+Y*Q)))
630     Phi=.5*EXP(-Y*Y)*P/Q
640     CASE <26.6
650     P=2.97886562639399289+Y*(7.40974060596474179+Y*(6.16020985310963054+Y*
(5.01904972678426746+Y*(1.27536664472996595+Y*.564189583547755074))))
660     Q=3.36907520698275277+Y*(9.60896532719278787+Y*(17.0814407474660043+Y*
(12.0489519278551290+Y*(9.39603401623505415+Y*(2.26052852076732697+Y)))))
670     Phi=.5*EXP(-Y*Y)*P/Q
680     CASE ELSE
690     Phi=0.
700     END SELECT
710     IF X>0. THEN Phi=1.-Phi
720     RETURN Phi
730     FNEND
740     !
750     SUB Hermite(DOUBLE N,REAL X,He(*))     !    He/n(X)                     eq. 50
760     DOUBLE K
770     He(0)=1.
780     He(1)=X
790     FOR K=2 TO N
800     He(K)=X*He(K-1)-(K-1)*He(K-2)
810     NEXT K
820     SUBEND
830     !
840     SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))                  !  eq. A-6
850     DOUBLE K,J
860     REAL Mom0
870     Mom(0)=Mom0=EXP(Cum(0))
880     FOR K=1 TO M
890     T=1.
900     S=Cum(K)*Mom0
910     FOR J=1 TO K-1
920     T=T*(K-J)/J
930     S=S+T*Cum(K-J)*Mom(J)
940     NEXT J
950     Mom(K)=S
960     NEXT K
970     SUBEND
980     !
```

PROGRAM PHRC (cont'd)

```
 990   SUB Cumnt_via_momnt(DOUBLE M,REAL Mom(*),Cum(*))          !    eq. A-7
1000   DOUBLE K,J
1010   REAL Mom0
1020   Mom0=Mom(0)
1030   Cum(0)=LOG(Mom0)
1040   FOR K=1 TO M
1050   T=1.
1060   S=Mom(K)
1070   FOR J=1 TO K-1
1080   T=T*(K-J)/J
1090   S=S-T*Mom(J)*Cum(K-J)
1100   NEXT J
1110   Cum(K)=S/Mom0
1120   NEXT K
1130   SUBEND
1140   !
1150   SUB Coeffhr_via_cum(DOUBLE M,REAL Alpha,Beta,Cum(*),A(*))
1160   ALLOCATE B(0:M)
1170   DOUBLE K,J,Mx
1180   F=Beta*Beta
1190   Cum(1)=(Cum(1)-Alpha)/Beta       !  MODIFIED NORMALIZED
1200   Cum(2)=Cum(2)/F-1.               !  CUMULANTS FOR K=1 & 2; eq. 63
1210   FOR K=3 TO M
1220   F=F*Beta*(K-1)
1230   Cum(K)=Cum(K)/F                  !  NORMALIZED CUMULANTS; eq. 62
1240   NEXT K
1250   A(0)=B(0)=EXP(Cum(0))
1260   F=1.
1270   FOR K=1 TO M
1280   S=0.
1290   FOR J=1 TO K
1300   S=S+Cum(J)*A(K-J)
1310   NEXT J
1320   A(K)=S/K
1330   F=F*K
1340   B(K)=A(K)*SQR(F)
1350   NEXT K
1360   Mx=Mx+10
1370   IF Mx<M THEN 1360
1380   Threshold=-7.
1390   T2=Threshold*2.
1400   V=10.^Threshold
1410   GINIT
1420   PLOTTER IS "GRAPHICS"
1430   GRAPHICS ON
1440   WINDOW 0.,FLT(Mx),T2,0.
1450   LINE TYPE 3
```

PROGRAM PHRC (cont'd)

```
1460    FOR J=0 TO Mx STEP 10
1470    MOVE J,T2
1480    DRAW J,0.
1490    NEXT J
1500    FOR J=T2 TO 0
1510    MOVE 0.,J
1520    DRAW Mx,J
1530    NEXT J
1540    PENUP
1550    LINE TYPE 1
1560    IMAGE 4D,2(4X,M.17DE)
1570    PRINT "    K                    B(K)                                Sum"
1580    Sum=0.
1590    FOR K=0 TO M
1600    B=B(K)
1610    Sum=Sum+B*B
1620    PRINT USING 1560;K,B,Sum
1630    IF B<V THEN 1660
1640    Y=LGT(B)
1650    GOTO 1700
1660    IF B>-V THEN 1690
1670    Y=T2-LGT(-B)
1680    GOTO 1700
1690    Y=Threshold
1700    PLOT K,Y
1710    NEXT K
1720    PENUP
1730    SUBEND
1740    !
1750    SUB Cumulants(DOUBLE M,REAL P0,Cum(*))    !  SHOT NOISE   eqs. 147-150
1760    Overlap=6.2               !  AV. NO. PULSES/SEC * AVERAGE PULSE DURATION
1770    Sigmaa=1.                 !  PARAMETER OF RAYLEIGH AMPLITUDE PDF
1780    P0=EXP(-Overlap)          !  PROBABILITY OF ZERO AMPLITUDE OF SHOT NOISE
1790    ALLOCATE Mom(0:M)         !  ARRAY FOR MOMENTS
1800    DOUBLE K
1810    S=Sigmaa*Sigmaa
1820    Cum(0)=0.
1830    Cum(1)=Overlap*Sigmaa*.25*PI*SQR(.5*PI)
1840    Cum(2)=Overlap*S*4./3.
1850    FOR K=3 TO M
1860    Cum(K)=Cum(K-2)*S*K*K/(K+1)
1870    NEXT K
1880    CALL Momnt_via_cumnt(M,Cum(*),Mom(*))
1890    Mom(0)=Mom(0)-P0          !  MOMENT CORRECTION FOR IMPULSE AT ORIGIN
1900    CALL Cumnt_via_momnt(M,Mom(*),Cum(*))
1910    SUBEND
```

PROGRAM pHRC

```
10 !   CONTINUOUS PART OF SHOT NOISE PDF, pc(u)               TR 7377, FIGURE 9
20 !   COEFFICIENTS OF HERMITE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
30     M=80    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
40     DOUBLE M,I,N,K                       !  INTEGERS < 2^31 = 2,147,483,648
50     REDIM Cum(0:M),A(0:M),He(0:M)
60     REAL Cum(0:100),A(0:100),He(0:100)
70     CALL Cumulants(M,P0,Cum(*))          !  P0 IS STEP AT ORIGIN
80     Center=Cum(1)                        !  CENTER OF PDF pc(u)
90     R2=Cum(2)                            !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                          !  RMS SPREAD OF pc(u)
110    Alpha0=Center                        !  THE CHOICES   Alpha=Alpha0   AND
120    Beta0=Rms                            !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=Center
140      Beta=Rms*1.5
150    CALL Coeffhr_via_cum(M,Alpha,Beta,Cum(*),A(*))     !    RC
160    PRINT "Center = ";Center
170    PRINT "Rms =";Rms
180    F1=1./(Beta*SQR(2.*PI))
190      INPUT "ORDER AND LIMITS:",N,U1,U2
200    PRINT "ORDER AND LIMITS:",N;U1;U2
210    Du=(U2-U1)/100.
220    PLOTTER IS "GRAPHICS"
230    GRAPHICS ON
240    WINDOW U1,U2,0.,.15
250    GRID 6.,.03
260    FOR I=0 TO 100
270    U=U1+Du*I
280    T=(U-Alpha)/Beta
290    CALL Hermite(N,T,He(*))
300    Sum=A(0)
310    FOR K=1 TO N
320    Sum=Sum+A(K)*He(K)
330    NEXT K
340    P=F1*EXP(-.5*T*T)*Sum                !    PDF OF RV AT U
350    PLOT U,P
360    NEXT I
370    PENUP
380    GOTO 190
390    END
400  ! USE ROUTINES IN PHRC
```

PROGRAM PHDMandRM

```
 10 ! STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u);  COEFFICIENTS OF
 20 ! HERMITE EXPANSION FOUND DIRECTLY VIA MOMENTS OR RECURSIVELY VIA MOMENTS
 30   M=80    !   MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
 40   DOUBLE M,I,N,K                     !  INTEGERS < 2^31 = 2,147,483,648
 50   REDIM Mom(0:M),A(0:M),He(0:M)
 60   REAL Mom(0:100),A(0:100),He(0:100),P(0:100)
 70   CALL Moments(M,P0,Mom(*))          !  P0 IS STEP AT ORIGIN
 80   Center=Mom(1)/Mom(0)               !  CENTER OF PDF pc(u)
 90   R2=Mom(2)/Mom(0)-Center*Center     !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                        !  RMS SPREAD OF pc(u)
110   Alpha0=Center                      !  THE CHOICES   Alpha=Alpha0   AND
120   Beta0=Rms                          !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=Center
140     Beta=Rms*1.5
150   CALL Coeffhd_via_mom(M,Alpha,Beta,Mom(*),A(*))     !    DM
160 ! CALL Coeffhr_via_mom(M,Alpha,Beta,Mom(*),A(*))     !    RM
170   PRINT "Center = ";Center
180   PRINT "Rms =";Rms
190   F1=1./SQR(2.*PI)
200     INPUT "ORDER AND LIMITS:",N,U1,U2
210   PRINT "ORDER AND LIMITS:",N;U1;U2
220   Du=(U2-U1)/100.
230   PLOTTER IS "GRAPHICS"
240   GRAPHICS ON
250   WINDOW U1,U2,-10.,0.
260   GRID Du*10.,1.
270   FOR I=0 TO 100
280   U=U1+Du*I
290   T=(U-Alpha)/Beta
300   CALL Hermite(N,T,He(*))
310   Sum=0.
320   FOR K=1 TO N
330   Sum=Sum+A(K)*He(K-1)
340   NEXT K
350   P=A(0)*FNPhi(T)-F1*EXP(-.5*T*T)*Sum     !   PROBABILITY THAT RV < U
360   IF U>=0. THEN P=P+P0                    !   ADDITION OF STEP AT ORIGIN
370   P(I)=P
380   IF P>0. THEN 410
390   PENUP
400   GOTO 420
410   PLOT U,LGT(P)
420   NEXT I
430   PENUP
440   FOR I=0 TO 100
450   U=U1+Du*I
460   P1=1.-P(I)
470   IF P1>0. THEN 500
480   PENUP
490   GOTO 510
500   PLOT U,LGT(P1)
510   NEXT I
520   PENUP
530   GOTO 200
540   END
550   !
```

PROGRAM PHDMandRM (cont'd)

```
560    DEF FNPhi(X)          !  HART, page 140, #5708 & #5725
570  ! LISTED IN PHRC
740    FNEND
750    !
760    SUB Hermite(DOUBLE N,REAL X,He(*))     !   He/n(X)
770  ! LISTED IN PHRC
830    SUBEND
840    !
850    SUB Hermite_i(DOUBLE N,REAL X,Hi(*)) ! Hi/n(X)=(-i)^n He/n(iX) eq.74-5
860    DOUBLE K                            !    MODIFIED HERMITE POLYNOMIALS
870    Hi(0)=1.
880    Hi(1)=X
890    FOR K=2 TO N
900    Hi(K)=X*Hi(K-1)+(K-1)*Hi(K-2)
910    NEXT K
920    SUBEND
930    !
940    SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
950  ! LISTED IN PHRC
1070   SUBEND
1080   !
1090   SUB Coeffhd_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1100   ALLOCATE He(0:M),F(0:M),B(0:M)
1110   DOUBLE K,J,Mx
1120   CALL Hermite(M,-Alpha/Beta,He(*))
1130   T=F(0)=1.
1140   FOR K=1 TO M
1150   F=F(K)=F(K-1)*K
1160   T=T*Beta
1170   He(K)=He(K)/F             !   NORMALIZED HERMITE POLYNOMIALS; eq. 68
1180   Mom(K)=Mom(K)/(F*T)       !   NORMALIZED MOMENTS re Beta; eq. 69
1190   NEXT K
1200   FOR K=0 TO M
1210   S=0.
1220   FOR J=0 TO K
1230   S=S+He(J)*Mom(K-J)
1240   NEXT J
1250   A(K)=S
1260   NEXT K
1270   MAT F=SQR(F)
1280   MAT B=A.F
1290   Mx=Mx+10
1300   IF Mx<M THEN 1290
1310   Threshold=-7.
1320   T2=Threshold*2.
1330   V=10.^Threshold
1340   GINIT
1350   PLOTTER IS "GRAPHICS"
1360   GRAPHICS ON
1370   WINDOW 0.,FLT(Mx),T2,0.
1380   LINE TYPE 3
```

PROGRAM PHDMandRM (cont'd)

```
1390    FOR J=0 TO Mx STEP 10
1400    MOVE J,T2
1410    DRAW J,0.
1420    NEXT J
1430    FOR J=T2 TO 0
1440    MOVE 0.,J
1450    DRAW Mx,J
1460    NEXT J
1470    PENUP
1480    LINE TYPE 1
1490    IMAGE 4D,2(4X,M.17DE)
1500    PRINT "   K                B(K)                            Sum"
1510    Sum=0.
1520    FOR K=0 TO M
1530    B=B(K)
1540    Sum=Sum+B*B
1550    PRINT USING 1490;K,B,Sum
1560    IF B<V THEN 1590
1570    Y=LGT(B)
1580    GOTO 1630
1590    IF B>-V THEN 1620
1600    Y=T2-LGT(-B)
1610    GOTO 1630
1620    Y=Threshold
1630    PLOT K,Y
1640    NEXT K
1650    PENUP
1660    SUBEND
1670    !
1680    SUB Coeffhr_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1690    ALLOCATE Hi(0:M),F(0:M),B(0:M)
1700    DOUBLE K,J,Mx
1710    CALL Hermite_i(M,Alpha/Beta,Hi(*))
1720    T=F(0)=1.
1730    FOR K=1 TO M
1740    F=F(K)=F(K-1)*K
1750    T=T*Beta
1760    Hi(K)=Hi(K)/F  ! NORMALIZED MODIFIED HERMITE POLYNOMIALS; eqs. 80 & 74
1770    Mom(K)=Mom(K)/(F*T)        !   NORMALIZED MOMENTS re Beta; eq. 69
1780    NEXT K
1790    FOR K=0 TO M
1800    S=Mom(K)
1810    FOR J=1 TO K
1820    S=S-Hi(J)*A(K-J)
1830    NEXT J
1840    A(K)=S
1850    NEXT K
1860    MAT F=SQR(F)
1870    MAT B=A.F
```

PROGRAM PHDMandRM (cont'd)

```
1880    Mx=Mx+10
1890    IF Mx<M THEN 1880
1900    Threshold=-7.
1910    T2=Threshold*2.
1920    V=10.^Threshold
1930    GINIT
1940    PLOTTER IS "GRAPHICS"
1950    GRAPHICS ON
1960    WINDOW 0.,FLT(Mx),T2,0.
1970    LINE TYPE 3
1980    FOR J=0 TO Mx STEP 10
1990    MOVE J,T2
2000    DRAW J,0.
2010    NEXT J
2020    FOR J=T2 TO 0
2030    MOVE 0.,J
2040    DRAW Mx,J
2050    NEXT J
2060    PENUP
2070    LINE TYPE 1
2080    IMAGE 4D,2(4X,M.17DE)
2090    PRINT "    K                  B(K)                          Sum"
2100    Sum=0.
2110    FOR K=0 TO M
2120    B=B(K)
2130    Sum=Sum+B*B
2140    PRINT USING 2080;K,B,Sum
2150    IF B<V THEN 2180
2160    Y=LGT(B)
2170    GOTO 2220
2180    IF B>-V THEN 2210
2190    Y=T2-LGT(-B)
2200    GOTO 2220
2210    Y=Threshold
2220    PLOT K,Y
2230    NEXT K
2240    PENUP
2250    SUBEND
2260    !
2270    SUB Moments(DOUBLE M,REAL P0,Cum(*))    !  SHOT NOISE    eqs. 147-150
2280    Overlap=6.2              !  AV. NO. PULSES/SEC * AVERAGE PULSE DURATION
2290    Sigmaa=1.                !  PARAMETER OF RAYLEIGH AMPLITUDE PDF
2300    P0=EXP(-Overlap)         !  PROBABILITY OF ZERO AMPLITUDE OF SHOT NOISE
2310    ALLOCATE Cum(0:M)        !  ARRAY FOR CUMULANTS
2320    DOUBLE K
2330    S=Sigmaa*Sigmaa
2340    Cum(0)=0.
2350    Cum(1)=Overlap*Sigmaa*.25*PI*SQR(.5*PI)
2360    Cum(2)=Overlap*S*4./3.
2370    FOR K=3 TO M
2380    Cum(K)=Cum(K-2)*S*K*K/(K+1)
2390    NEXT K
2400    CALL Momnt_via_cumnt(M,Cum(*),Mom(*))
2410    Mom(0)=Mom(0)-P0         !  MOMENT CORRECTION FOR IMPULSE AT ORIGIN
2420    SUBEND
```

PROGRAM pHDMandRM

```
10 !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u); COEFFICIENTS OF HERMITE
20 !  EXPANSION FOUND DIRECTLY VIA MOMENTS OR RECURSIVELY VIA MOMENTS
30    M=80     !   MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40    DOUBLE M,I,N,K                        !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Mom(0:M),A(0:M),He(0:M)
60    REAL Mom(0:100),A(0:100),He(0:100)
70    CALL Moments(M,P0,Mom(*))             !  P0 IS STEP AT ORIGIN
80    Center=Mom(1)/Mom(0)                  !  CENTER OF PDF pc(u)
90    R2=Mom(2)/Mom(0)-Center*Center        !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                           !  RMS SPREAD OF pc(u)
110   Alpha0=Center                         !  THE CHOICES   Alpha=Alpha0   AND
120   Beta0=Rms                             !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=Center
140     Beta=Rms*1.5
150   CALL Coeffhd_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   DM
160 ! CALL Coeffhr_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   RM
170   PRINT "Center = ";Center
180   PRINT "Rms =";Rms
190   F1=1./(Beta*SQR(2.*PI))
200     INPUT "ORDER AND LIMITS:",N,U1,U2
210   PRINT "ORDER AND LIMITS:",N;U1;U2
220   Du=(U2-U1)/100.
230   PLOTTER IS "GRAPHICS"
240   GRAPHICS ON
250   WINDOW U1,U2,0.,.15
260   GRID 6.,.03
270   FOR I=0 TO 100
280   U=U1+Du*I
290   T=(U-Alpha)/Beta
300   CALL Hermite(N,T,He(*))
310   Sum=A(0)
320   FOR K=1 TO N
330   Sum=Sum+A(K)*He(K)
340   NEXT K
350   P=F1*EXP(-.5*T*T)*Sum                       !     PDF OF RV AT U
360   PLOT U,P
370   NEXT I
380   PENUP
390   GOTO 200
400   END
410 ! USE ROUTINES IN PHDM&RM
```

PROGRAM PLRC

```
10 ! STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u);  TR 7377, FIGURE 20
20 ! COEFFICIENTS OF GEN. LAGUERRE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
30    M=70    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Cum(0:M),A(0:M),L(0:M)
60    REAL Cum(0:100),A(0:100),L(0:100),P(0:100)
70    CALL Cumulants(M,P0,Cum(*))       !  P0 IS STEP AT ORIGIN
80    Center=Cum(1)                     !  CENTER OF PDF pc(u)
90    R2=Cum(2)                         !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110   Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0    AND
120   Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=.74
140     Beta=2.1
150   CALL Coefflr_via_cum(M,Alpha,Beta,Cum(*),A(*))     !   RC
160   PRINT "Center = ";Center
170   PRINT "Rms =";Rms
180   A1=Alpha+1.
190   O1=1./A1
200   F1=1./FNGamma(A1)
210     INPUT "ORDER AND LIMITS:",N,U1,U2
220   PRINT "ORDER AND LIMITS:",N;U1;U2
230   Du=(U2-U1)/100.
240   PLOTTER IS "GRAPHICS"
250   GRAPHICS ON
260   WINDOW U1,U2,-11.,0.
270   GRID 4.,1.
280   P(0)=P0
290   PLOT 0.,LGT(P0)
300   FOR I=1 TO 100
310   U=U1+Du*I
320   T=U/Beta
330   CALL Laguerre(N-1,A1,T,L(*))
340   Sum=A(0)*FNF1(A1,T)*O1
350   FOR K=1 TO N
360   Sum=Sum+A(K)*L(K-1)/K
370   NEXT K
380   P(I)=P=P0+F1*EXP(-T+A1*LOG(T))*Sum    !  PROBABILITY THAT RV < U
390   IF P>0. THEN 420
400   PENUP
410   GOTO 430
420   PLOT U,LGT(P)
430   NEXT I
440   PENUP
450   FOR I=0 TO 100
460   U=U1+Du*I
470   P1=1.-P(I)
480   IF P1>0. THEN 510
490   PENUP
500   GOTO 520
510   PLOT U,LGT(P1)
520   NEXT I
530   PENUP
540   GOTO 210
550   END
560   !
```

PROGRAM PLRC (cont'd)

```
570     DEF FNGamma(X)   !  Gamma(X) via HART, page 282, #5243               eq. 2
580     DOUBLE N,K
590     N=INT(X)
600     R=X-N
610     IF N>0 OR R<>0. THEN 640
620     PRINT "FNGamma(X) IS NOT DEFINED FOR X = ";X
630     STOP
640     IF R>0. THEN 670
650     Gamma2=1.
660     GOTO 740
670     P=439.33044406002567 6+R*(50.108693752970953 0+R*6.7449507245925289 9)
680     P=8762.7102978521489 6+R*(2008.5274013072791 2+R*P)
690     P=42353.689509744089 6+R*(20886.861789269887 4+R*P)
700     Q=499.02852662143904 8-R*(189.49823415702801 6-R*(23.081551524580125-R))
710     Q=9940.3074150827709 0-R*(1528.6072737795220 2+R*Q)
720     Q=42353.689509744090 0+R*(2980.3853309256649 9-R*Q)
730     Gamma2=P/Q                        !  Gamma(2+R) for 0 < R < 1
740     IF N>2 THEN 780
750     IF N<2 THEN 830
760     Gamma=Gamma2
770     RETURN Gamma
780     Gamma=Gamma2
790     FOR K=1 TO N-2
800     Gamma=Gamma*(X-K)
810     NEXT K
820     RETURN Gamma
830     R=1.
840     FOR K=0 TO 1-N
850     R=R*(X+K)
860     NEXT K
870     Gamma=Gamma2/R
880     RETURN Gamma
890     FNEND
900     !
910     DEF FNF1(A1,X)                    !  1F1(1;A1+1;X)                  eq. C-2
920     DOUBLE K
930     T=S=1.
940     FOR K=1 TO 200
950     T=T*X/(A1+K)
960     S=S+T
970     IF T<=1.E-17*S THEN RETURN S
980     NEXT K
990     PRINT "200 TERMS IN FNF1 AT";A1;X
1000    RETURN S
1010    FNEND
1020    !
```

PROGRAM PLRC (cont'd)

```
1030    SUB Laguerre(DOUBLE N,REAL Alpha,X,L(*))                        !   eq. 96
1040    DOUBLE K
1050    A1=Alpha-1.
1060    L(0)=1.
1070    L(1)=Alpha+1.-X
1080    FOR K=2 TO N
1090    L(K)=((K+K+A1-X)*L(K-1)-(K+A1)*L(K-2))/K
1100    NEXT K
1110    SUBEND
1120    !
1130    SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
1140  ! LISTED IN PHRC
1260    SUBEND
1270    !
1280    SUB Cumnt_via_momnt(DOUBLE M,REAL Mom(*),Cum(*))
1290  ! LISTED IN PHRC
1420    SUBEND
1430    !
1440    SUB Coefflr_via_cum(DOUBLE M,REAL Alpha,Beta,Cum(*),A(*))
1450    ALLOCATE B(0:M),C(0:M),D(1:M)
1460    DOUBLE K,J,J1,Mx
1470    T=Beta
1480    Cum(1)=Cum(1)/T
1490    FOR K=2 TO M
1500    T=T*Beta*(K-1)
1510    Cum(K)=Cum(K)/T            !   NORMALIZED CUMULANTS; eq. 62
1520    NEXT K
1530    A1=Alpha+1.
1540    FOR J=1 TO M
1550    J1=J+1
1560    T=1.
1570    S=A1
1580    FOR K=1 TO J
1590    T=T*(K-J1)/K
1600    S=S+T*Cum(K)
1610    NEXT K
1620    D(J)=S
1630    NEXT J
1640    A(0)=B(0)=C(0)=EXP(Cum(0))
1650    Q=1.
1660    FOR K=1 TO M
1670    S=0.
1680    FOR J=1 TO K
1690    S=S+D(J)*C(K-J)
1700    NEXT J
1710    C(K)=C=S/K
1720    Q=Q*K/(Alpha+K)
1730    A(K)=C*Q
1740    B(K)=C*SQR(Q)
1750    NEXT K
```

PROGRAM PLRC (cont'd)

```
1760    Mx=Mx+10
1770    IF Mx<M THEN 1760
1780    Threshold=-7.
1790    T2=Threshold*2.
1800    V=10.^Threshold
1810    GINIT
1820    PLOTTER IS "GRAPHICS"
1830    GRAPHICS ON
1840    WINDOW 0.,FLT(Mx),T2,0.
1850    LINE TYPE 3
1860    FOR J=0 TO Mx STEP 10
1870    MOVE J,T2
1880    DRAW J,0.
1890    NEXT J
1900    FOR J=T2 TO 0
1910    MOVE 0.,J
1920    DRAW Mx,J
1930    NEXT J
1940    PENUP
1950    LINE TYPE 1
1960    IMAGE 4D,2(4X,M.17DE)
1970    PRINT "    K                    B(K)                                     Sum"
1980    Sum=0.
1990    FOR K=0 TO M
2000    B=B(K)
2010    Sum=Sum+B*B
2020    PRINT USING 1960;K,B,Sum
2030    IF B<V THEN 2060
2040    Y=LGT(B)
2050    GOTO 2100
2060    IF B>-V THEN 2090
2070    Y=T2-LGT(-B)
2080    GOTO 2100
2090    Y=Threshold
2100    PLOT K,Y
2110    NEXT K
2120    PENUP
2130    SUBEND
2140    !
2150    SUB Cumulants(DOUBLE M,REAL P0,Cum(*))       !  SHOT NOISE
2160  ! LISTED IN PHRC
2310    SUBEND
```

PROGRAM pLRC

```
10  !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u)              TR 7377, FIGURE 21
20  !  COEFFS. OF GENERAL. LAGUERRE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
30     M=70    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
40     DOUBLE M,I,N,K                          !  INTEGERS < 2^31 = 2,147,483,648
50     REDIM Cum(0:M),A(0:M),L(0:M)
60     REAL Cum(0:100),A(0:100),L(0:100)
70     CALL Cumulants(M,P0,Cum(*))            !  P0 IS STEP AT ORIGIN
80     Center=Cum(1)                          !  CENTER OF PDF pc(u)
90     R2=Cum(2)                              !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                            !  RMS SPREAD OF pc(u)
110    Alpha0=Center*Center/R2-1.             !  THE CHOICES   Alpha=Alpha0   AND
120    Beta0=R2/Center                        !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=.74
140      Beta=2.1
150    CALL Coefflr_via_cum(M,Alpha,Beta,Cum(*),A(*))     !    RC
160    PRINT "Center = ";Center
170    PRINT "Rms =";Rms
180    F1=1./(Beta*FNGamma(Alpha+1.))
190      INPUT "ORDER AND LIMITS:",N,U1,U2
200    PRINT "ORDER AND LIMITS:",N;U1;U2
210    Du=(U2-U1)/100.
220    PLOTTER IS "GRAPHICS"
230    GRAPHICS ON
240    WINDOW U1,U2,0.,.15
250    GRID 6.,.03
260    FOR I=0 TO 100
270    U=U1+Du*I
280    IF U<0. THEN 400
290    IF U>0. THEN 320
300    PLOT 0.,0.
310    GOTO 400
320    T=U/Beta
330    CALL Laguerre(N,Alpha,T,L(*))
340    Sum=A(0)
350    FOR K=1 TO N
360    Sum=Sum+A(K)*L(K)
370    NEXT K
380    P=F1*EXP(-T+Alpha*LOG(T))*Sum    !    PDF OF RV AT U
390    PLOT U,P
400    NEXT I
410    PENUP
420    GOTO 190
430    END
440  ! USE ROUTINES IN PLRC
```

PROGRAM PLDMandRM

```
10 !  STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u);    COEFFICIENTS OF
20 !  GENERALIZED LAGUERRE EXPAN. FOUND DIRECTLY AND RECURSIVELY VIA MOMENTS
30    M=70   !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40    DOUBLE M,I,N,K                   !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Mom(0:M),A(0:M),L(0:M)
60    REAL Mom(0:100),A(0:100),L(0:100),P(0:100)
70    CALL Moments(M,P0,Mom(*))        !  P0 IS STEP AT ORIGIN
80    Center=Mom(1)/Mom(0)             !  CENTER OF PDF pc(u)
90    R2=Mom(2)/Mom(0)-Center*Center   !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                      !  RMS SPREAD OF pc(u)
110   Alpha0=Center*Center/R2-1.       !  THE CHOICES   Alpha=Alpha0    AND
120   Beta0=R2/Center                  !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=.74
140     Beta=2.1
150   CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*))    !   DM
160 ! CALL Coefflr_via_mom(M,Alpha,Beta,Mom(*),A(*))    !   RM
170   PRINT "Center = ";Center
180   PRINT "Rms =";Rms
190   A1=Alpha+1.
200   O1=1./A1
210   F1=1./FNGamma(A1)
220     INPUT "ORDER AND LIMITS:",N,U1,U2
230   PRINT "ORDER AND LIMITS:",N;U1;U2
240   Du=(U2-U1)/100.
250   PLOTTER IS "GRAPHICS"
260   GRAPHICS ON
270   WINDOW U1,U2,-11.,0.
280   GRID 4.,1.
290   P(0)=P0
300   PLOT 0.,LGT(P0)
310   FOR I=1 TO 100
320   U=U1+Du*I
330   T=U/Beta
340   CALL Laguerre(N-1,A1,T,L(*))
350   Sum=A(0)*FNF1(A1,T)*O1
360   FOR K=1 TO N
370   Sum=Sum+A(K)*L(K-1)/K
380   NEXT K
390   P(I)=P=P0+F1*EXP(-T+A1*LOG(T))*Sum    !   PROBABILITY THAT RV < U
400   IF P>0. THEN 430
410   PENUP
420   GOTO 440
430   PLOT U,LGT(P)
440   NEXT I
450   PENUP
460   FOR I=0 TO 100
470   U=U1+Du*I
480   P1=1.-P(I)
490   IF P1>0. THEN 520
500   PENUP
510   GOTO 530
520   PLOT U,LGT(P1)
530   NEXT I
540   PENUP
550   GOTO 220
560   END
570   !
```

PROGRAM PLDMandRC (cont'd)

```
 580    DEF FNGamma(X)   !   Gamma(X) via HART, page 282, #5243
 590  ! LISTED IN PLRC
 900    FNEND
 910    !
 920    DEF FNF1(A1,X)                       !   1F1(1;A1+1;X)
 930  ! LISTED IN PLRC
1020    FNEND
1030    !
1040    SUB Laguerre(DOUBLE N,REAL Alpha,X,L(*))   !   Ln\alpha(X)
1050  ! LISTED IN PLRC
1120    SUBEND
1130    !
1140    SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
1150  ! LISTED IN PLRC
1270    SUBEND
1280    !
1290    SUB Coeffld_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1300    ALLOCATE B(0:M)
1310    DOUBLE K,K1,J,Mx
1320    T=1.
1330    FOR K=1 TO M
1340    T=T*(Alpha+K)*Beta                    !   NORMALIZED MOMENTS re
1350    Mom(K)=Mom(K)/T                       !   Alpha and Beta; eq. 118
1360    NEXT K
1370    Q=1.
1380    A(0)=B(0)=Mom(0)
1390    FOR K=1 TO M
1400    K1=K+1
1410    T=1.
1420    S=Mom(0)
1430    FOR J=1 TO K
1440    T=T*(J-K1)/J
1450    S=S+T*Mom(J)
1460    NEXT J
1470    Q=Q*(Alpha+K)/K
1480    A(K)=S
1490    B(K)=S*SQR(Q)
1500    NEXT K
1510    Mx=Mx+10
1520    IF Mx<M THEN 1510
1530    Threshold=-7.
1540    T2=Threshold*2.
1550    V=10.^Threshold
1560    GINIT
1570    PLOTTER IS "GRAPHICS"
1580    GRAPHICS ON
1590    WINDOW 0.,FLT(Mx),T2,0.
1600    LINE TYPE 3
```

PROGRAM PLDMandRM (cont'd)

```
1610    FOR J=0 TO Mx STEP 10
1620    MOVE J,T2
1630    DRAW J,0.
1640    NEXT J
1650    FOR J=T2 TO 0
1660    MOVE 0.,J
1670    DRAW Mx,J
1680    NEXT J
1690    PENUP
1700    LINE TYPE 1
1710    IMAGE 4D,2(4X,M.17DE)
1720    PRINT "   K                  B(K)                         Sum"
1730    Sum=0.
1740    FOR K=0 TO M
1750    B=B(K)
1760    Sum=Sum+B*B
1770    PRINT USING 1710;K,B,Sum
1780    IF B<V THEN 1810
1790    Y=LGT(B)
1800    GOTO 1850
1810    IF B>-V THEN 1840
1820    Y=T2-LGT(-B)
1830    GOTO 1850
1840    Y=Threshold
1850    PLOT K,Y
1860    NEXT K
1870    PENUP
1880    SUBEND
1890    !
1900    SUB Coefflr_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1910    ALLOCATE B(0:M)
1920    DOUBLE K,K1,J,Mx
1930    T=1.
1940    FOR K=1 TO M
1950    T=T*(Alpha+K)*Beta                  !  NORMALIZED MOMENTS re
1960    Mom(K)=Mom(K)/T                     !  Alpha and Beta; eq. 118
1970    NEXT K
1980    Q=1.
1990    A(0)=B(0)=Mom(0)
2000    FOR K=1 TO M
2010    K1=K+1
2020    T=1.
2030    S=Mom(K)-A(0)
2040    FOR J=1 TO K-1
2050    T=T*(J-K1)/J
2060    S=S-T*A(J)
2070    NEXT J
2080    IF K MOD 2=1 THEN S=-S
2090    A(K)=S
2100    Q=Q*(Alpha+K)/K
2110    B(K)=S*SQR(Q)
2120    NEXT K
```

PROGRAM PLDMandRM (cont'd)

```
2130    Mx=Mx+10
2140    IF Mx<M THEN 2130
2150    Threshold=-7.
2160    T2=Threshold*2.
2170    V=10.^Threshold
2180    GINIT
2190    PLOTTER IS "GRAPHICS"
2200    GRAPHICS ON
2210    WINDOW 0.,FLT(Mx),T2,0.
2220    LINE TYPE 3
2230    FOR J=0 TO Mx STEP 10
2240    MOVE J,T2
2250    DRAW J,0.
2260    NEXT J
2270    FOR J=T2 TO 0
2280    MOVE 0.,J
2290    DRAW Mx,J
2300    NEXT J
2310    PENUP
2320    LINE TYPE 1
2330    IMAGE 4D,2(4X,M.17DE)
2340    PRINT "    K                  B(K)                          Sum"
2350    Sum=0.
2360    FOR K=0 TO M
2370    B=B(K)
2380    Sum=Sum+B*B
2390    PRINT USING 2330;K,B,Sum
2400    IF B<V THEN 2430
2410    Y=LGT(B)
2420    GOTO 2470
2430    IF B>-V THEN 2460
2440    Y=T2-LGT(-B)
2450    GOTO 2470
2460    Y=Threshold
2470    PLOT K,Y
2480    NEXT K
2490    PENUP
2500    SUBEND
2510    !
2520    SUB Moments(DOUBLE M,REAL P0,Mom(*))        !  SHOT NOISE
2530  ! LISTED IN PHDM&RM
2670    SUBEND
```

PROGRAM pLDMandRM

```
 10 !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u); COEFFICIENTS OF GENERALIZED
 20 !  LAGUERRE EXPANSION FOUND DIRECTLY AND RECURSIVELY VIA MOMENTS
 30    M=70    !    MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
 40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
 50    REDIM Mom(0:M),A(0:M),L(0:M)
 60    REAL Mom(0:100),A(0:100),L(3:100)
 70    CALL Moments(M,P0,Mom(*))         !  P0 IS STEP AT ORIGIN
 80    Center=Mom(1)/Mom(0)              !  CENTER OF PDF pc(u)
 90    R2=Mom(2)/Mom(0)-Center*Center    !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110    Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0    AND
120    Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=.74
140      Beta=2.1
150    CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*))      !   DM
160  ! CALL Coefflr_via_mom(M,Alpha,Beta,Mom(*),A(*))      !   RM
170    PRINT "Center = ";Center
180    PRINT "Rms =";Rms
190    F1=1./(Beta*FNGamma(Alpha+1.))
200      INPUT "ORDER AND LIMITS:",N,U1,U2
210    PRINT "ORDER AND LIMITS:",N;U1;U2
220    Du=(U2-U1)/100.
230    PLOTTER IS "GRAPHICS"
240    GRAPHICS ON
250    WINDOW U1,U2,0.,.15
260    GRID 6.,.03
270    PLOT 0.,0.
280    FOR I=1 TO 100
290    U=U1+Du*I
300    T=U/Beta
310    CALL Laguerre(N,Alpha,T,L(*))
320    Sum=A(0)
330    FOR K=1 TO N
340    Sum=Sum+A(K)*L(K)
350    NEXT K
360    P=F1*EXP(-T+Alpha*LOG(T))*Sum     !   PDF OF RV AT U
370    PLOT U,P
380    NEXT I
390    PENUP
400    GOTO 200
410    END
420  ! USE ROUTINES IN PLDM&RM
```

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